Basics on The Theory of Ends

Eduardo Ventilari Sodré

February 2024

1 Definition of Ends

Throughout the rest of the text, let X be a Hausdorff, connected, locally connected, locally compact and second countable topological space. For most applications, X can be taken as a manifold, possibly with boundary.

If X is not compact, then we can describe the Alexandroff extension (one point compactification) of X. This is done by adjoining to X a point ∞ and a topology generated by the open sets of X along with the sets $(X \setminus K) \cup \{\infty\}$, for all compact sets $K \subset X$. The resulting space $\widehat{X} = X \cup \{\infty\}$ will be compact Hausdorff and have X as a dense open subset. This encapsulates the intuitive idea that for a sequence to "diverge" to infinity, it must eventually leave every compact set of X. However, we can make a more precise description of the possible ways a sequence may diverge to infinity in X. This is done by constructing the ends $\mathcal{E}(X)$ of X.

For $K \subseteq L$ compact subsets of X, we have a natural inclusion $X \setminus L \subseteq X \setminus K$. This gives us a map

$$\iota_{KL} : \mathrm{Comp}(X \setminus L) \to \mathrm{Comp}(X \setminus K)$$

from the set of (connected) components of $X \setminus L$ to the set of components of $X \setminus K$, since each $U \in \mathrm{Comp}(X \setminus L)$, being connected, will be contained in a unique component of $X \setminus K$. Naturally, for $K \subseteq L \subseteq M$ compact sets, we have $\iota_{KL} \circ \iota_{LM} = \iota_{KM}$.

This gives us an inverse system $(\mathrm{Comp}(X \setminus K), \iota_{KL})_K$ running over all compact subsets of X. By embuing each set $\mathrm{Comp}(X \setminus K)$ with the discrete topology, we define the set of *ends* $\mathcal{E}(X)$ of X the be the inverse limit of this system:

$$\begin{array}{ccccccc} & & & \mathcal{E}(X) & & & \\ & & \swarrow^{\iota_L} & & \searrow^{\iota_K} & & \\ \cdots \longrightarrow & \mathrm{Comp}(X \setminus L) & & \xrightarrow{\quad \iota_{KL} \quad} & & \mathrm{Comp}(X \setminus K) & \longrightarrow \cdots \end{array}$$

This inverse limit, which always exists in the category of topological spaces, comes equipped with continuous maps $\iota_K : \mathcal{E}(X) \to \mathrm{Comp}(X \setminus K)$ that make the inverse system commute: $\iota_K = \iota_{KL} \circ \iota_L$. The topology on $\mathcal{E}(X)$ is generated by the sets

$$\widehat{U} = \{e \in \mathcal{E}(X) \colon \iota_K(e) = U\} = \iota_K^{-1}(U),$$

running over all compact sets $K \subseteq X$ and all components U of $X \setminus K$. This is the coarsets topology on $\mathcal{E}(X)$ which makes the maps ι_K continuous. Note that possibly $\mathcal{E}(X) = \emptyset$; this will be the case if X is compact, since $\mathrm{Comp}(X \setminus X) = \emptyset$ will be in the inverse system, and $\mathcal{E}(X)$ must map to it.

Intuitively, for an end $e \in \mathcal{E}(X)$, $\iota_K(e)$ represents the component of $X \setminus K$ which contains this end, being a neighborhood of it. In fact, we may give the space $X \sqcup \mathcal{E}(X)$ a natural topology, generated by the open sets of X and by the sets of the form $U \cup \iota_K^{-1}(U)$ for $U \in \mathrm{Comp}(X \setminus K)$. We will return to this topology later.

Though very general and independent of additional constructions in X, this definition by an inverse limit is somewhat hard to work with. Hence we introduce another description of $\mathcal{E}(X)$ which, historically, was how the theory was first described.

An *exhaustion by compact sets* of X is a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that, for all $n \in \mathbb{N}$, $K_n \subseteq \operatorname{int} K_{n+1}$ and $X = \bigcup_{n\in\mathbb{N}} K_n$. Under the given hypotheses, X always admits a compact exhaustion. An important property of compact exhaustions is that for every compact $K \subseteq X$, there is some $n \in \mathbb{N}$ such that $K \subseteq K_n$; this is because $\{\operatorname{int} K_n\}_n$ will be an open cover of K, admitting a finite subcover. In some contexts it is required that each K_n be connected, but for the theory that follows this is not necessary.

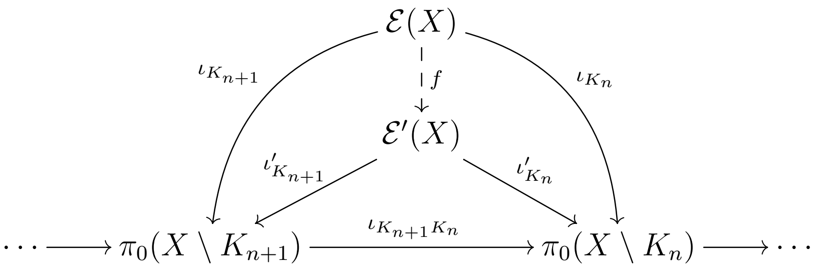
Given a compact exhaustion of X, we may consider the inverse system generated by it:

$$\cdots \to \pi_0(X \setminus K_{n+1}) \to \pi_0(X \setminus K_n) \to \pi_0(K_{n-1}) \to \cdots \to \pi_0(X \setminus K_0).$$

Let $\mathcal{E}'(X)$ be its inverse limit. As such a sequence is *cofinal* in the full inverse system of compact sets, we obtain a unique identification between $\mathcal{E}'(X)$ and $\mathcal{E}(X)$:

Proposition 1.1. *The spaces $\mathcal{E}(X)$ and $\mathcal{E}'(X)$ are canonically homeomorphic.*

Proof. Let $\iota'_{K_n} : \mathcal{E}'(X) \to \mathrm{Comp}(X \setminus K_n)$ be the maps given by the inverse limit $\mathcal{E}'(X)$. As we also have the maps $\iota_{K_n} : \mathcal{E}(X) \to \mathrm{Comp}(X \setminus K_n)$, by the universal property of the inverse limit, we have a continuous map $f : \mathcal{E}(X) \to \mathcal{E}'(X)$ that makes the diagram commute:



Now, since $(K_n)_n$ is a compact exhaustion, for any compact set $K \subseteq X$ there exists some sufficiently big n such that $K \subseteq K_n$. Hence we may take the map $\iota_{K_n K} \circ \iota'_{K_n} : \mathcal{E}'(X) \to \pi_0(X \setminus K)$, which will not depend on the choice of n due to commutativity of the diagrams. With this, we have by the universal property of the inverse limit, a unique continuous map $g : \mathcal{E}'(X) \to \mathcal{E}(X)$ that makes all of the diagrams commute. Again by the universal property, we must have that $g \circ f$ and $f \circ g$ are the identity maps. This implies that for any choice of compact exhaustion of X, we have a unique canonical isomorphism from $\mathcal{E}(X)$ to the space of ends of this exhaustion. $\square$

With this, we have a fairly concrete description of and end $e \in \mathcal{E}(X)$: given a compact exhaustion of X, e will be uniquely determined by a sequence $(U_n)_n$ of connected open components of $X \setminus K_n$ such that

$$\cdots \subseteq U_{n+2} \subseteq U_{n+1} \subseteq U_n \subseteq \cdots \subseteq U_1 \subseteq U_0.$$

In particular, the characterization of the ends will not depend on the particular exhaustion by compact sets we use.

2 Results

It is not immediately obvious that, for example, if X is non-compact, then $\mathcal{E}(X) \neq \emptyset$. This is because even though we may find sequences of points diverging off to infinity, it is hard to keep track of which components of the complements of compact sets the points belong to. On a similar note, it is not obvious that $\mathcal{E}(X)$ is compact or that $X \subseteq X \sqcup \mathcal{E}(X)$ is a compactification of X, since the sets $\mathrm{Comp}(X \setminus K)$ may be infinite. These results will indeed be true under the hypotheses we have made on X, notably local connectedness and local compactness.

We start with the following lemma, which essentially says that if you fatten a compact set, all but finitely many components of its complement will be covered:

Lemma 2.1. *Let $K, L \subseteq X$ be nonempty compact subsets such that $K \subseteq \operatorname{int} L$. Then for all but finitely many components U of $X \setminus K$ we have that $U \subseteq \operatorname{int} L$.*

Proof. As X is locally connected, each component of $X \setminus K$ is open. Moreover, for U a component, $\partial U \subseteq K$; otherwise there would be some $x \in \partial U \setminus K$, and x is contained in a component V of $X \setminus K$. But any neighborhood of x must intersect U, so $U = V$ by connectedness. But then $x \in U \cap \partial U$, which cannot happen as U is open.

Suppose, for the sake of contradiction, that there exist infinitely many distinct components $\{U_i\}_{i \in \mathbb{N}}$ of $X \setminus K$ such that $U_i \nsubseteq L$, so that $U_i \cap L^c$ is open and nonempty. Let M be a compact set in X such that $L \subseteq \operatorname{int} M$; M always exists because X is locally compact. Explicitly, take a compact neighborhood of each $x \in L$, so that a finite subcover will be a compact set which contains L in its interior.

We claim that for all $i \in \mathbb{N}$, we have that $U_i \cap L^c \cap \operatorname{int} M \neq \emptyset$. For if $U_i \cap L^c \cap \operatorname{int} M = \emptyset$, then

$$U_i \subseteq (L^c \cap \operatorname{int} M)^c = L \cup (\operatorname{int} M)^c.$$

Note that L and $(\operatorname{int} M)^c$ are disjoint closed subsets of X. As U_i is connected, we must have that either $U_i \subseteq L$ or $U_i \subseteq (\operatorname{int} M)^c$. By hypothesis, the first situation does not happen, so $U_i \cap \operatorname{int} M = \emptyset$, and in particular, $U_i \cap L = \emptyset$. Because $K \neq \emptyset$, U_i is nonempty and not the whole X, so there exists $y_i \in \partial U_i \subseteq K$. As $K \subseteq \operatorname{int} L$, any sufficiently small neighborhood of y_i is contained in L, and since $y_i \in \partial U_i$, this neighborhood must intersect U_i, a contradiction with $U_i \cap L = \emptyset$. Hence the claim is shown.

For each $i \in \mathbb{N}$, take $z_i \in U_i \cap L^c \cap \operatorname{int} M$. (This uses countable choice.) As M is compact and X is first countable, M is sequentially compact, so $(z_i)_i$ has a convergent subsequence $z_{i_k} \to z_\infty$. As $z_{i_k} \notin L$ for all k and $K \subseteq \operatorname{int} L$, we have that $z_\infty \notin K$. Let W be the component of $X \setminus K$ to which z_∞ belongs. As it is a neighborhood of z_∞, for sufficiently large k we have that $z_{i_k} \in W$. But z_{i_k} belongs to the component U_{i_k}, so $U_{i_k} = W$; as the components are distinct for all $i \in \mathbb{N}$, this cannot happen. Therefore for all but finitely many $U \in \operatorname{Comp}(X \setminus K)$, we have $U \subseteq L$, and as U is open, $U \subseteq \operatorname{int} L$. □

Theorem 2.2. *For $K \subseteq X$ a compact set, $X \setminus K$ has at most finitely many non-precompact components. Moreover, if $\mathcal{P} \subseteq \operatorname{Comp}(X \setminus K)$ denotes the set of precompact components of $X \setminus K$, then $K \cup \bigcup \mathcal{P}$ is compact.*

Intuitively, this theorem states that by filling in all of the holes of a compact set (the precompact components), we still have a compact set.

Proof. Consider a compact set $L \subseteq X$ such that $K \subseteq \operatorname{int} L$. Every non-precompact component of $X \setminus K$ cannot be contained in L, hence by the previous lemma there must be finitely many of them.

Note that $K \cup \bigcup \mathcal{P}$ is closed because its complement is the union of the non-precompact components of $X \setminus K$, which are open. Let $U_1, \ldots, U_m$ be those precompact components of $X \setminus K$ that are not contained in L. Then $L \cup \overline{U_1} \cup \cdots \cup \overline{U_m}$ is compact and contains $K \cup \bigcup \mathcal{P}$, so that the latter is compact. □

A compact set is *full* if its complement has no precompact components (intituvely, it has no holes). The previous theorem gives us a way of attributing to any compact set $K \subseteq X$ its *hull* $\operatorname{Hull}(K)$, which is the union of K wth all of its precompact components. Naturally, $\operatorname{Hull}(K)$ is full, compact and contains K, and $\operatorname{Hull}(\operatorname{Hull}(K)) = \operatorname{Hull}(K)$.

Proposition 2.3. *Given $K \subseteq X$ compact, if $L \subseteq X$ is a full compact set such that $K \subseteq L$, then* $\operatorname{Hull}(K) \subseteq L$.

Proof. If $\operatorname{Hull}(K) \not\subseteq L$, then there exists a precompact component U of $X \setminus K$ and some $x \in U \setminus L \neq \emptyset$. Then x belongs to some component V of $X \setminus L$, and since $x \in U$, we have that $\iota_{KL}(V) = U$, that is, $V \subseteq U$. But as L is full, all components of $X \setminus L$ are not precompact, so $V \subseteq U$ precompact is a contradiction. □

The above proposition also shows that $K \subseteq L \implies \operatorname{Hull}(K) \subseteq \operatorname{Hull}(L)$; this is because $K \subseteq L \subseteq \operatorname{Hull}(L)$, and as $\operatorname{Hull}(L)$ is full, $\operatorname{Hull}(K) \subseteq \operatorname{Hull}(L)$.

The corollary below, of fundamental importance, is exercise 3.3.4 in [2]:

Corollary 2.4. *X admits an exhaustion by full compact sets.*

Proof. Given a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of X, let $L_n = \operatorname{Hull}(K_n)$. As $(K_n)_n$ is a compact exhaustion, there exists some n_1 such that $L_0 \subseteq \operatorname{int} K_{n_1} \subseteq \operatorname{int} L_{n_1}$. Similarly, there will exist $n_2 > n_1$ such that $L_{n_1} \subseteq \operatorname{int} K_{n_2} \subseteq \operatorname{int} L_{n_2}$. Proceeding by induction, we create a sequence $(L_{n_m})_m$ of full compact sets forming an exhaustion. □

By considering a full compact exhaustion of X, each set $\operatorname{Comp}(X \setminus K_n)$ will be finite, consisting of the non-precompact components. This allows us to prove the following:

Theorem 2.5. *$\mathcal{E}(X)$ is Hausdorff, compact and second countable, and if $\mathcal{E}(X) \neq \emptyset$, then it is also totally disconnected.*

Proof. As an inverse limit of discrete finite topological spaces, $\mathcal{E}(X)$ will be a subset of the product $\prod \mathrm{Comp}(X \setminus K_n)$, which is Hausdorff and compact by Tychonoff's theorem. That $\mathcal{E}(X)$ is a closed subset is a consequence of the fact that $\mathcal{E}(X)$ is an intersection of closed sets

$$\mathcal{E}(X) = \bigcap_{i,j} \{\mathbf{x} \in \prod \mathrm{Comp}(X \setminus K_n) \colon \iota_{ij} \circ \pi_j(\mathbf{x}) = \pi_i(\mathbf{x})\},$$

so $\mathcal{E}(X)$ is compact.

If $\mathcal{E}(X) \neq \emptyset$ and $C \subseteq \mathcal{E}(X)$ is nonempty and connected, then $\iota_n(C)$ is connected for all n. But the connected nonempty sets of $\mathrm{Comp}(X \setminus K_n)$ are the singletons, so for all n there exists a unique $U_n \in \mathrm{Comp}(X \setminus K_n)$ such that $\iota_n(C) = \{U_n\}$. All ends in C therefore define the same sequence, hence must represent the same end; so C is a singleton. □

Corollary 2.6. *X is compact if and only if $\mathcal{E}(X) = \emptyset$.*

Proof. We have already seen that if X is compact then $\mathcal{E}(X) = \emptyset$. Now suppose X is not compact. Then, given an exhaustion $(K_n)_n$ of X by full compact sets, we have that every set $\mathrm{Comp}(X \setminus K_n)$ is finite non-empty.

Pick any $U_0 \in \mathrm{Comp}(X \setminus K_0)$. As U_0 is not precompact, $U_0 \not\subseteq K_1$, so that there exists some $x_1 \in U_0 \setminus K_1$. This point must belong to some component $U_1 \in \mathrm{Comp}(X \setminus K_1)$, and $U_1 \subseteq U_0$. We may proceed inductively by picking $x_n \in U_n \setminus K_{n+1}$ and considering its component, which amounts to verifying that the preimage $\iota_{n,n+1}^{-1}(U_n)$ is not empty. The sequence $(U_n)_{n \in \mathbb{N}}$ thus constructed will be an end in $\mathcal{E}(X)$. □

The above proof shows that for a non-compact space X and a full compact exhaustion of it, all the maps in the inverse system will be surjective. In particular, the number of components of $X \setminus K_n$ is non-decreasing with n.

3 The Freudenthal Compactification

We say that a set $U \subseteq X$ is a neighborhood of an end $e \in \mathcal{E}(X)$ if U contains $\iota_K(e)$, for some $K \subseteq X$ compact. This definition agrees with the topology on $X \sqcup \mathcal{E}(X)$.

Proposition 3.1. *If U is the neighborhood of an end, then U is not precompact.*

Proof. Consider a full compact exhaustion $(K_n)_n$ of X. If U contains $\iota_K(e)$, then it will also contain $\iota_{K_n}(e)$ for some n. As all components of $X \setminus K_n$ are not precompact, U is not precompact. □

The converse does not necessarily hold, even if we require that U be connected: consider a sector of angle $0 < \theta < 2\pi$ around 0 in $\mathbb{C} \setminus \{0\}$ and radius 1. However, it holds if U is a component of $X \setminus K$ for some K compact:

Proposition 3.2. *If $K \subseteq X$ is compact and U is a component $X \setminus K$, then U is the neighborhood of an end if and only if U is not precompact.*

Proof. Suppose U is not precompact. Consider a full compact exhaustion $(K_n)_n$ of X such that $K \subseteq K_0$. As in the proof of 2.6, there is a component U_0 of $X \setminus K_0$ such that $U_0 \subseteq U$, and starting from it, we may construct a sequence of components $(U_n)_n$ representing an end e of which U is a neighborhood. □

Recall that we imbue $X \sqcup \mathcal{E}(X)$ with the topology generated by the open sets of X and the sets of the form

$$U \cup \iota_K^{-1}(U),$$

running over all compact sets $K \subseteq X$ and $U \in \text{Comp}(X \setminus K)$. We also have topological embeddings $X \hookrightarrow X \sqcup \mathcal{E}(X)$ and $\mathcal{E}(X) \hookrightarrow X \sqcup \mathcal{E}(X)$ given by the inclusions, where X is open and $\mathcal{E}(X)$ is closed. Moreover, X will be a dense open set in $X \sqcup \mathcal{E}(X)$.

Theorem 3.3. *$X \sqcup \mathcal{E}(X)$ is Hausdorff, second countable, compact, connected and locally connected.*

Proof. Given that X and $\mathcal{E}(X)$ are Hausdorff, we need only find separating neighborhoods for $x \in X$ and $e \in \mathcal{E}(X)$. As X is locally compact, we may find $x \in U \subseteq K$ where U is open and K is compact, so that U and $(X \setminus K) \cup \mathcal{E}(X)$ are disjoint open sets separating x and e. $X \sqcup \mathcal{E}(X)$ is second countable by considering a countable base for X and a full compact exhaustion of X.

Suppose that $(x_k)_{k\in\mathbb{N}} \subset X$ is a sequence diverging to infinity, and consider a full compact exhaustion $(K_n)_n$ of X. Since each set $X \setminus K_0$ has finitely many components, there exists $U_0 \in \text{Comp}(X \setminus K_0)$ such that infinitely many of the x_k are contained in U_0. This gives us a subsequence $(x_k^0)_k$ of $(x_k)_k$. Consider now $\iota_{0,1}^{-1}(U_0) \subseteq \text{Comp}(X \setminus K_1)$, which is nonempty by the arguments in 2.6 and consists of finitely many components which infinitely many of the x_k^0 belong to. Hence we may find $U_1 \in \iota_{0,1}^{-1}(U_0)$ and a subsequence $(x_k^1)_k$ such that $x_k^1 \in U_1$ for all k. By proceeding inductively, we obtain a sequence

$$U_n \subseteq \cdots \subseteq U_1 \subseteq U_0$$

of components of $X \setminus K_n$ and subsequences $(x_k^m)_k$ for $m \leq n$ such that, for $m \leq n$ and all k, $x_k^m \in U_m$.

By taking the diagonal subsequence $(y_n)_n = (x_n^n)_n$, we have an infinite sequence

$$\cdots \subseteq U_n \subseteq \cdots \subseteq U_1 \subseteq U_0$$

and $y_n \in U_n$ for all n. Hence this sequence converges to the end determined by the decreasing sequence of components. This concludes that $X \sqcup \mathcal{E}(X)$ is sequentially compact, hence compact.

As X is connected, its closure in $X \sqcup \mathcal{E}(X)$ is connected. But every end is the limit of some sequence of points in X, by choosing $x_n \in \iota_{K_n}(e)$ in a compact exhaustion of X. Hence $X \sqcup \mathcal{E}(X)$ is connected.

Finally, we show local connectivity at each end by showing that it admits a basis of connected neighborhoods. Given a compact exhaustion of X, a basis of neighborhoods for $e \in \mathcal{E}(X)$ is given by

$$\iota_{K_n}(e) \cup \iota_{K_n}^{-1}(\iota_{K_n}(e)).$$

Each $U_n = \iota_{K_n}(e)$ is connected, So its closure in $X \sqcup \mathcal{E}(X)$ is also connected. As $U_n \cup \iota_{K_n}^{-1}(U_n)$ is contained in the closure of U_n by the reasonings above and contains U_n, it is connected. □

As a useful criterion for convergence, we see that a sequence $(x_n)_n \subset X$ converges to an end $e \in \mathcal{E}(X)$ if and only if, for all compact sets $K \subseteq X$, there exists some $N \in \mathbb{N}$ such that, for $n \geq N$, $x_n \in \iota_K(e)$, where we recall that $\iota_K(e)$ is the component of $X \setminus K$ which "contains" the end, being a neighborhood of it.

If X is non-compact and $\widehat{X}$ denotes the one point compactification of X, we construct a map $f : X \sqcup \mathcal{E}(X) \to \widehat{X}$, which is the identity on X and, for all $e \in \mathcal{E}(X)$, $f(e) = \infty$. This map is continuous; if $U \subseteq \widehat{X}$ is open, then either $\infty \notin U$, in which case $f^{-1}(U) = U \subseteq X$, or $\infty \in U$, where then $K = X \setminus (U \setminus \{\infty\})$ is compact. then

$$f^{-1}(U) = \mathcal{E}(X) \cup (X \setminus K).$$

Since the topology on $X \sqcup \mathcal{E}(X)$ is generated by the subsets

$$U \sqcup \iota_K^{-1}(U) = U \cup \{e \in \mathcal{E}(X) \colon \iota_K(e) = U\}$$

for all compact sets K and components U of $X \setminus K$, by taking the union of these sets over all components $U \in \pi_0(X \setminus K)$ we obtain $\mathcal{E}(X) \cup (X \setminus K)$. By the previous results, we also have that f is a proper map.

4 Proper Maps

Let X and Y be two spaces having the basic hypotheses assumed in the introduction. We now wish to understand continuous proper maps $f : X \to Y$, that is, continuous maps for which the preimages of compact sets are compact. They are the appropriate class of maps to study behavior at infinity; intuitively, proper maps preserve diverging sequences, and they act at infinity predictably. Recall that as Y is locally compact Hausdorff, we also have that f proper implies it is a closed map. (We will assume that the proper maps are continuous throughout.)

Suppose f is proper. Then, for a compact set $L \subseteq Y$, $f^{-1}(L)$ is compact, and $f^{-1}(Y \setminus L) = X \setminus f^{-1}(L)$. Taking preimages preserves the inclusions, and we also have a map

$$f_* : \mathrm{Comp}(X \setminus f^{-1}(L)) \to \mathrm{Comp}(Y \setminus L)$$

taking U to the component of $Y \setminus L$ that contains $f(U)$, as $f(U)$ is connected. If $L \subseteq L'$ are compact, this map will make the diagram commute:

$$\begin{array}{ccc}
\mathrm{Comp}(X \setminus f^{-1}(L')) & \xrightarrow{\iota_{f^{-1}(L)f^{-1}(L')}} & \mathrm{Comp}(X \setminus f^{-1}(L)) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
\mathrm{Comp}(Y \setminus L') & \xrightarrow{\iota_{LL'}} & \mathrm{Comp}(Y \setminus L)
\end{array}$$

More generally, f_* will map from the inverse system of X to the inverse system of Y. By considering the composition

$$f_* \circ \iota_{f^{-1}(L)} : \mathcal{E}(X) \to \mathrm{Comp}(Y \setminus L),$$

and by the universal property of the inverse limit, there exists a unique continuous map from $f_* : \mathcal{E}(X) \to \mathcal{E}(Y)$ that makes the diagrams commute.

Explicitly, since

$$\begin{array}{ccc}
\mathcal{E}(X) & \xrightarrow{f_*} & \mathcal{E}(Y) \\
{\scriptstyle \iota_{f^{-1}(L)}}\downarrow & & \downarrow{\scriptstyle \iota_L} \\
\mathrm{Comp}(X \setminus f^{-1}(L)) & \xrightarrow{f_*} & \mathrm{Comp}(X \setminus L)
\end{array}$$

commutes, for an end $e \in \mathcal{E}(X)$, $f_*(e)$ will be the unique end of Y such that all neighborhoods of e are mapped into neighborhoods of $f_*(e)$.

This will serve us to extend the proper map $f : X \to Y$ to the ends of the spaces:

Theorem 4.1. *Let $f : X \to Y$ be a proper map. Then there exists a unique continuous extension $\widehat{f} : X \sqcup \mathcal{E}(X) \to Y \sqcup \mathcal{E}(Y)$.*

Proof. Define the map $\widehat{f}$ to be f on X and f_* on $\mathcal{E}(X)$. It will be continuous, since, for any open set $V \subseteq Y$, $\widehat{f}^{-1}(V) = f^{-1}(V)$ is open, and for any open set of the form $V \cup \iota_L^{-1}(V)$ for $L \subseteq Y$ compact and V a component of $Y \setminus L$, we have

$$\begin{aligned}\widehat{f}^{-1}(V \cup \iota_L^{-1}(V)) &= f^{-1}(V) \cup f_*^{-1}(\iota_L^{-1}(V)) = f^{-1}(V) \cup (\iota_L \circ f_*)^{-1}(V) \\ &= f^{-1}(V) \cup (f_* \circ \iota_{f^{-1}(L)})^{-1}(V) = f^{-1}(V) \cup \iota_{f^{-1}(L)}^{-1}((f_*^{-1}(V)),\end{aligned}$$

where $f_*^{-1}(V)$ consists of all the components of $X \setminus f^{-1}(L)$ that get mapped into V. Hence

$$f^{-1}(V) \cup \iota_{f^{-1}(L)}^{-1}((f_*^{-1}(V)) = f^{-1}(V) \cup \bigcup_{f_*(U)=V} \iota_{f^{-1}(L)}^{-1}(U).$$

Additionally, $f^{-1}(V) = \bigcup_{f_*(U)=V} U$; that $f^{-1}(V) \supseteq \bigcup_{f_*(U)=V} U$ is evident, and if $x \in f^{-1}(V) \subseteq X \setminus f^{-1}(L)$, then x must belong to exactly one component U of $X \setminus f^{-1}(L)$. Since $f(x) \in V$, we have that $f(U) \cap V \neq \emptyset$, so that by connectedness $f(U) \subseteq V$ and $f_*(U) = V$. This shows that

$$f^{-1}(V) \cup \bigcup_{f_*(U)=V} \iota_{f^{-1}(L)}^{-1}(U) = \bigcup_{f_*(U)=V} (U \cup \iota_{f^{-1}(L)}^{-1}(U)),$$

which is a union of open sets in $X \sqcup \mathcal{E}(X)$, hence open.

We now show uniqueness of $\widehat{f}$. If $\tilde{f} : X \sqcup \mathcal{E}(X) \to \mathcal{E}(Y)$ is continuous extending f, then, for a sequence $(x_n)_n \subset X$ converging to $e \in \mathcal{E}(X)$, we must have that $(f(x_n))_n$ converges to $\tilde{f}(e)$. Let $L \subseteq Y$ be compact. Then for all sufficiently big n, $f(x_n) \in \iota_L(\tilde{f}(e))$. But since for all big n we have $x_n \in \iota_{f^{-1}(L)}(e)$, by connectedness we must have that the component $\iota_{f^{-1}(L)}(e)$ gets mapped into $\iota_L(\tilde{f}(e))$, so

$$f_*(\iota_{f^{-1}(L)}(e)) = \iota_L(\tilde{f}(e)).$$

as this is true for all ends $e \in \mathcal{E}(X)$ and all compact sets $L \subseteq Y$, and because $f_* : \mathcal{E}(X) \to \mathcal{E}(Y)$ is unique making the diagrams commute, we must have that $\tilde{f}|_{\mathcal{E}(X)} = f_*$, so $\tilde{f} = \widehat{f}$. □

Given a self-homeomorphism $f : X \to X$, because it is proper, it will have a unique continuous extension to the ends. It will in fact permute them:

Theorem 4.2. *Given a self-homeomorphism $f : X \to X$, the extension $\widehat{f} : X \sqcup \mathcal{E}(X) \to X \sqcup \mathcal{E}(X)$ is a homeomorphism.*

Proof. Consider the maps $f_* : \mathcal{E}(X) \to \mathcal{E}(X)$ and $(f^{-1})_* : \mathcal{E}(X) \to \mathcal{E}(X)$. We see that, for $K \subseteq X$ compact,

$$\begin{aligned}(f_* \circ (f^{-1})_*) \circ \iota_{f^{-1}(K)} &= f_* \circ (f^{-1})_* \circ \iota_{f^{-1}(K)} \\ = f_* \circ \iota_K \circ (f^{-1})_* &= \iota_{f^{-1}(K)} \circ f_* \circ (f^{-1})_* = \iota_{f^{-1}(K)},\end{aligned}$$

because the latter maps f_* and $(f^{-1})_*$ are acting on components and are therefore inverses of each other. As this is true for all K, and the identity map $\mathrm{id}_{\mathcal{E}(X)} : \mathcal{E}(X) \to \mathcal{E}(X)$ also satisfies this commutativity property, we must have that $f_* \circ (f^{-1})^* = \mathrm{id}_{\mathcal{E}(X)}$, and analogously $(f^{-1})_* \circ f_* = \mathrm{id}_{\mathcal{E}(X)}$. Hence f_* is a homeomorphism and $(f_*)^{-1} = (f^{-1})_*$. This glues with f and f^{-1} to show that $\widehat{f} \circ \widehat{f^{-1}} = \widehat{f^{-1}} \circ \widehat{f} = \mathrm{id}$. □

All of the results above are very categorical in nature, which stems from the fact that the inverse limit and the ends of a space are covariant functors in the class of topological spaces with proper maps.

5 Questions

Problem 1. Given a full compact exhaustion of X, is it easily shown that $|\mathcal{E}(X)| = \lim_{n\to\infty} |\operatorname{Comp}(X \setminus K_n)|$?

Problem 2. How do proper maps interact with full compact sets and hulls? Is the image/preimage of a full set full, or are there counterexamples?

Problem 3. Is there some universal property that the compactification $X \sqcup \mathcal{E}(X)$ satisfies, analogously to the universal property of the one point compactification $\widehat{X} = X \cup \{\infty\}$?

Problem 4. What do (sufficiently small) neighborhoods of and end look like? If X is a surface of finite genus and e an isolated end, will it have a neighborhood homeomorphic to $\mathbb{D} \setminus \{0\}$? By filling in the ends, do we still obtain a surface? Can this be used to classify finite genus surfaces with finite ends, or do the proofs require this classification result in the first place?

Problem 5. Given a proper map $r : [0, 1) \to X$, it will have a unique continuous extension mapping 1 to an end of X. How can we use this to identify ends? What would be the definition of a proper homotopy of proper rays? What purpose would it serve?

Problem 6. Given a concrete X, can we easily identify its ends, and what $X \sqcup \mathcal{E}(X)$ is? For punctured compact surfaces, this is easy enough; it is sufficient to find a "workable" exhaustion by full compact sets.

References

[1] G. Peschke, *The Theory of Ends*, Nieuw Archief voor Wiskunde **8** (1990), 1–12.

[2] C. R. Guilbault, *Ends, shapes, and boundaries in manifold topology and geometric group theory*, Arxiv (2021).

[3] H. Freudenthal, *Über die Enden topologischer Räume und Gruppen*, Mathematische Zeitschrift **33** (1931), 692–713.