A Glimpse on Holomorphic Dynamics

Graduate Student Seminar

Eduardo V. Sodré

Brown

September 2023

Newton's Method

Root finding algorithms: how to find the roots of a real function f?

Newton's Method

Root finding algorithms: how to find the roots of a real function f?

$f \in C^1$ (e.g. polynomial), *Newton's method*:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If $g(x) := x - f(x)/f'(x)$,

$$f(x) = 0 \iff g(x) = x.$$

Newton's Method

Root finding algorithms: how to find the roots of a real function f?

$f \in C^1$ (e.g. polynomial), *Newton's method*:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If $g(x) := x - f(x)/f'(x)$,

$$f(x) = 0 \iff g(x) = x.$$

Geometric meaning of g: take tangent line to f at $(x, f(x))$ and calculate its zero.

Most initial points converge to a zero of f!

Newton's Method

E.g.: finding $\sqrt{3}$. Take $f(x) = x^2 - 3$,

$$x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right).$$

Newton's Method

E.g.: finding $\sqrt{3}$. Take $f(x) = x^2 - 3$,

$$x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right).$$

Starting from $x_0 = 1$:

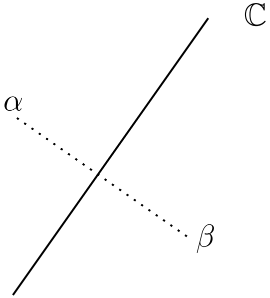
$$1 \mapsto 2 \mapsto 1.75 \mapsto 1.7321... \mapsto 1.7320508...$$

For general f, some inital points may take longer to converge, or seem to not converge at all.

Newton's Method

What happens in $\mathbb{C}$?

For $f(x) = (x - \alpha)(x - \beta)$, $\alpha \neq \beta$, $\mathbb{C}$ is bissected into halves: One converges to α, the other to β.



Newton's Method

What about more general f, say, $f(x) = x^3 - 1$?

Newton's Method

What about more general f, say, $f(x) = x^3 - 1$?

Idea: take many points, iterate many times, see where they end, and color them.

$$g(x) = \frac{2}{3}x - \frac{1}{3x^2},$$

Iteration of a rational function on $\mathbb{C}$.

“Newton’s” Fractal

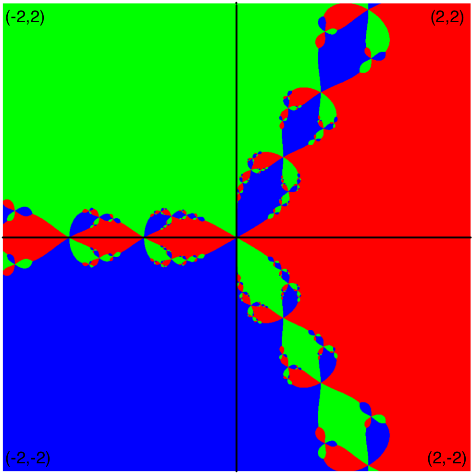


Figure: Newton’s Fractal

Global Dynamics

Endomorphisms of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: rational functions.

Global Dynamics

Endomorphisms of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: rational functions.

- What are the fixed points? And periodic points?

Global Dynamics

Endomorphisms of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: rational functions.

- What are the fixed points? And periodic points?
- Are they *attracting* or *repelling*? How are the dynamics near them?

Global Dynamics

Endomorphisms of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: rational functions.

- What are the fixed points? And periodic points?
- Are they *attracting* or *repelling*? How are the dynamics near them?
- What are the possible *orbits* of f?

Global Dynamics

Endomorphisms of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: rational functions.

- What are the fixed points? And periodic points?
- Are they *attracting* or *repelling*? How are the dynamics near them?
- What are the possible *orbits* of f?
- What are the limiting behaviors? Is it sensitive to initial conditions?

Global Dynamics

Endomorphisms of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$: rational functions.

- What are the fixed points? And periodic points?
- Are they *attracting* or *repelling*? How are the dynamics near them?
- What are the possible *orbits* of f?
- What are the limiting behaviors? Is it sensitive to initial conditions?
- What happens if we perturb f?

Fatou and Julia Sets

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be rational of degree $d \geq 2$.

$z \in \hat{\mathbb{C}}$ is in the **Fatou set** F_f if there exists some neighborhood U of z such that $\{f^n|_U\}_{n\in\mathbb{N}}$ is a normal family. That is, every sequence has subsequence converging locally uniformly.

Fatou and Julia Sets

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be rational of degree $d \geq 2$.

$z \in \hat{\mathbb{C}}$ is in the **Fatou set** F_f if there exists some neighborhood U of z such that $\{f^n|_U\}_{n \in \mathbb{N}}$ is a normal family. That is, every sequence has subsequence converging locally uniformly.

$J_f = \hat{\mathbb{C}} \setminus F_f$ is the **Julia set**.

Fatou and Julia Sets

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be rational of degree $d \geq 2$.

$z \in \hat{\mathbb{C}}$ is in the **Fatou set** F_f if there exists some neighborhood U of z such that $\{f^n|_U\}_{n \in \mathbb{N}}$ is a normal family. That is, every sequence has subsequence converging locally uniformly.

$J_f = \hat{\mathbb{C}} \setminus F_f$ is the **Julia set**.

Morally: nearby points in F_f have similar dynamics, and points in J_f display chaotic behavior: sensitive to initial conditions.

Fatou and Julia Sets

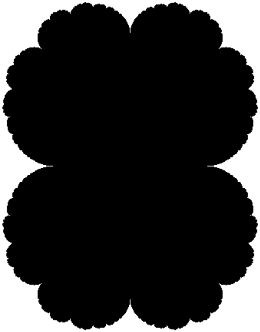


Figure: Julia set for $z^2 - \frac{1}{4}$

Fatou and Julia Sets

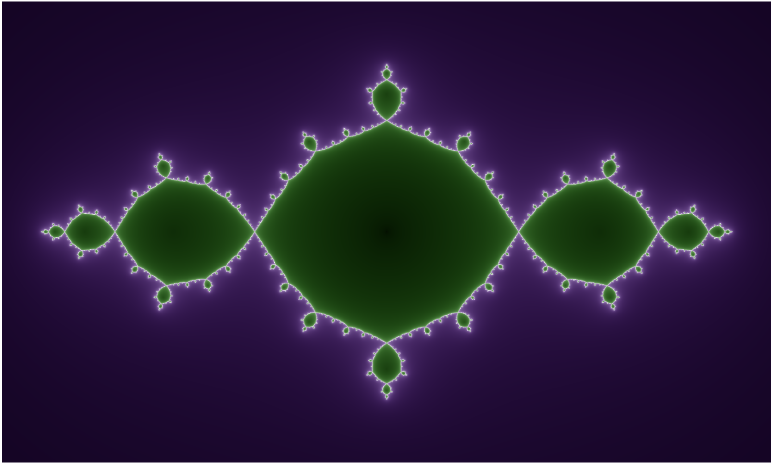


Figure: Julia set for $z^2 - 1$

Fatou and Julia Sets

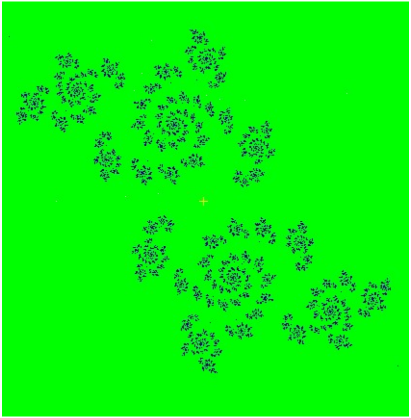


Figure: Julia set for $z^2 + (0.023 + 0.684i)$

Fatou and Julia Sets

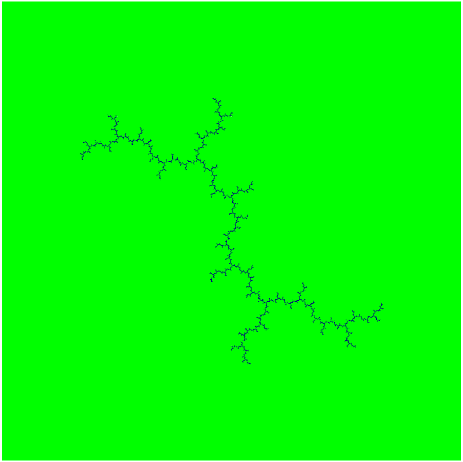


Figure: Julia set for $z^2 + i$

Fatou and Julia Sets

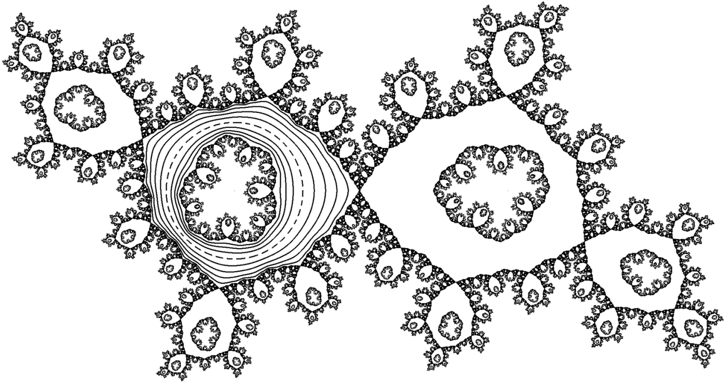


Figure: Julia set for a cubic rational map

Fatou and Julia Sets

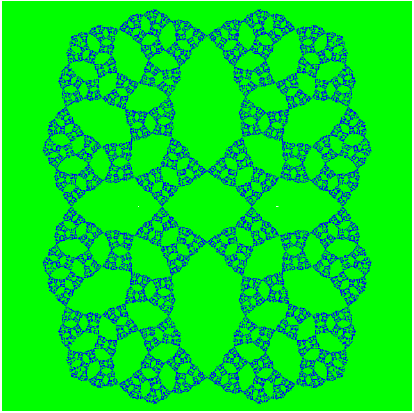


Figure: Julia set of $(z^2 - c)/(z^2 + c)$ for $c = -1.3$

Properties of Fatou and Julia Sets

Assume $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d \geq 2$.

- F_f is open and J_f is closed;
- F_f and J_f are totally invariant;
- $J_f \neq \emptyset$;
- For all $z \in J_f$, the preimages of z are dense in J_f;
- For generic $z \in J_f$, the orbit of z is dense in J_f;
- J_f is the closure of the repelling periodic orbits;
- If $\operatorname{int} J_f \neq \emptyset$, then $J_f = \hat{\mathbb{C}}$.

Polynomials

Case of $P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ polynomial: easier to describe.

Polynomials

Case of $P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ polynomial: easier to describe.

∞ is an attracting fixed pont: basin $\mathcal{A}(\infty)$, and $J = \partial\mathcal{A}(\infty)$.

Polynomials

Case of $P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ polynomial: easier to describe.

∞ is an attracting fixed pont: basin $\mathcal{A}(\infty)$, and $J = \partial \mathcal{A}(\infty)$.

Theorem (Fatou)

The Julia set J is connected if and only if for all critical points c of P $P^n(c) \not\to \infty$.

If, for some critical point c, $P^n(c) \to \infty$, then J_f has uncountably many connected components.

If, for all critical points c, $P^n(c) \to \infty$, then J is totally disconnected.

The Mandelbrot Set

Every quadratic polynomial $P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is conjugate to one of the form

$$P_c(z) = z^2 + c, \quad c \in \mathbb{C}.$$

The Mandelbrot Set

Every quadratic polynomial $P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is conjugate to one of the form

$$P_c(z) = z^2 + c, \quad c \in \mathbb{C}.$$

Same critical point 0 for all P_c. Locus of connectivity:

$$\begin{aligned} M &:= \{c \in \mathbb{C} \mid P_c^n(0) \not\to \infty\} \\ &= \{c \in \mathbb{C} \mid P_c^n(0) \text{ stays bounded } \} \\ &= \{c \in \mathbb{C} \mid J_{P_c} \text{ is connected } \} \end{aligned}$$

M is the **Mandelbrot Set**!

The Mandelbrot Set

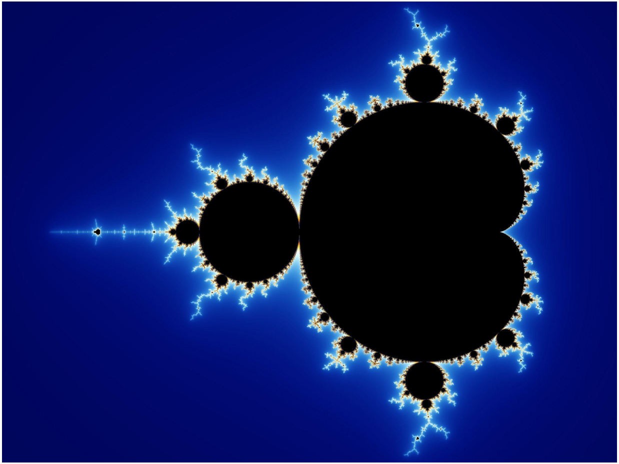
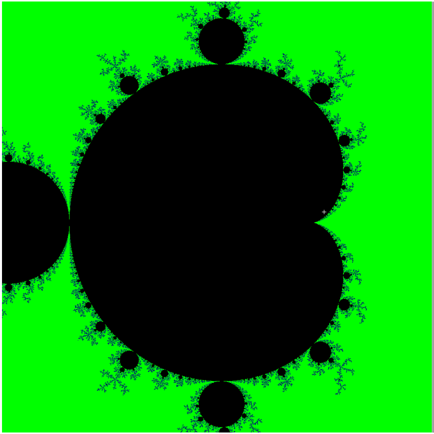
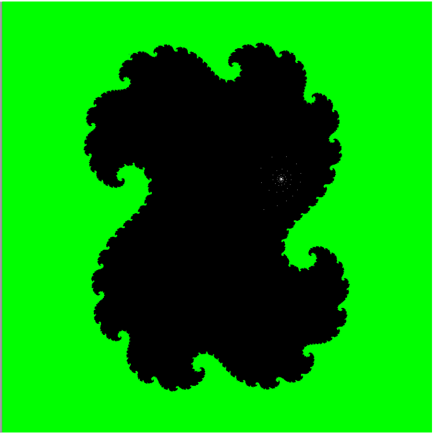


Figure: The Mandelbrot set

The Mandelbrot set

Main cardioid: $c \in M$ for which 0 converges to attracting fixed point.





The Mandelbrot Set

$c \in \mathbb{C}$ is a **hyperbolic parameter** if 0 converges to some attracting periodic cycle. In this case, $c \in \operatorname{int} M$.

The Mandelbrot Set

$c \in \mathbb{C}$ is a **hyperbolic parameter** if 0 converges to some attracting periodic cycle. In this case, $c \in \operatorname{int} M$.

Conjecture (Density of Hyperbolicity)

Every parameter $c \in \operatorname{int} M$ is hyperbolic.

The Mandelbrot Set

$c \in \mathbb{C}$ is a **hyperbolic parameter** if 0 converges to some attracting periodic cycle. In this case, $c \in \text{int}\, M$.

Conjecture (Density of Hyperbolicity)

Every parameter $c \in \text{int}\, M$ *is hyperbolic.*

Theorem (MLC $\implies$ DH)

If the Mandelbrot set is locally connected, every parameter $c \in \text{int}\, M$ *is hyperbolic.*

References

- Wolf Jung: https://mndynamics.com/indexp.html
- D. S. Alexander - *A History of Complex Dynamics From Schröder to Fatou and Julia*
- J. Milnor - *Dynamics in One Complex Variable*
- Adrien Douady & John H. Hubbard - *Exploring the Mandelbrot set. The Orsay notes*
- A. F. Beardon - *Iteration of Rational Functions Complex Analytic Dynamical Systems*