Hubbard trees, automata and ray connections

Early Career Complex Dynamics Workshop 2026

Eduardo Ventilari Sodré

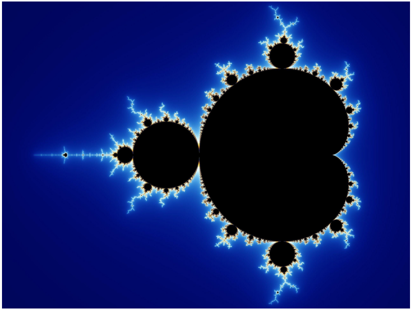
Brown University

June 2026

The Mandelbrot set

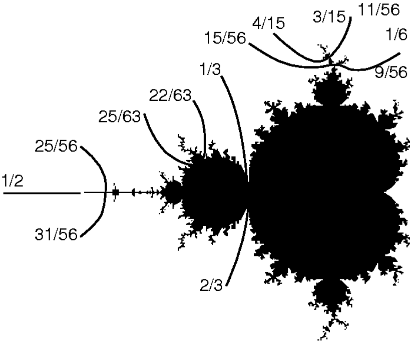
Mandelbrot set M: connectedness locus for quadratic polynomials.

$$M = \{c \in \mathbb{C} : J_{f_c} \text{ connected}, \ f_c(z) = z^2 + c\}$$



Parameter rays

- M is compact, connected and full
- Complement $\mathbb{C} \setminus M$ conformally isomorphic to $\mathbb{C} \setminus \overline{\mathbb{D}}$
- Can define **Parameter rays**:



PCF parameters

θ rational angle:

- Parameter ray $\mathcal{R}_M(\theta)$ lands at c_θ in the Mandelbrot set
- Julia set $J_\theta := J_{c_\theta}$ connected and locally connected
- $f_\theta(z) := z^2 + c_\theta$ postcritically finite
- Well defined **Hubbard tree** H_θ connecting the postcritical set

PCF parameters

θ rational angle:

- Parameter ray $\mathcal{R}_M(\theta)$ lands at c_θ in the Mandelbrot set
- Julia set $J_\theta := J_{c_\theta}$ connected and locally connected
- $f_\theta(z) := z^2 + c_\theta$ postcritically finite
- Well defined **Hubbard tree** H_θ connecting the postcritical set

Particular case of $\theta = a/2^m$ dyadic. Parameter ray lands at "tips of veins" of the Mandelbrot set.

$\theta = 1/4$

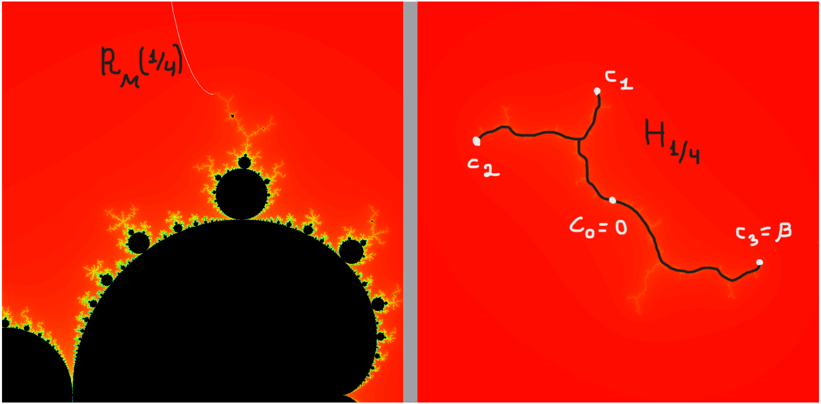


Figure: Left: parameter ray $R_M(1/4)$. Right: Hubbard tree $H_{1/4}$.

The main question

Because J_θ is locally connected, every external ray $\mathcal{R}(t)$ lands.

Given $\theta = a/2^m$, which external rays $\mathcal{R}(t)$ land on the Hubbard tree H_θ?

The main question

Because J_θ is locally connected, every external ray $\mathcal{R}(t)$ lands.

Given $\theta = a/2^m$, which external rays $\mathcal{R}(t)$ land on the Hubbard tree H_θ?

Relevance:

- Relationship of H_θ with *biacessibility*
- Rays that land together on H_θ form the *core lamination* of f_θ: monotonic along veins
- In matings of polynomials: to study long rays connections, sufficient to look for those passing through the Hubbard trees

The forbidden region

Describing the set of angles landing on H_θ:

The forbidden region

Describing the set of angles landing on H_θ:

- Find preimage Hubbard tree $H^{-1} := f^{-1}(H)$
- Find the "attaching points" of H^{-1} to H
- Find the angle sectors cutting of the branches of $H^{-1} \setminus H$

The forbidden region

Describing the set of angles landing on H_θ:

- Find preimage Hubbard tree $H^{-1} := f^{-1}(H)$
- Find the "attaching points" of H^{-1} to H
- Find the angle sectors cutting of the branches of $H^{-1} \setminus H$

Union of all sectors: **forbidden region** F.

$\theta = 1/4$

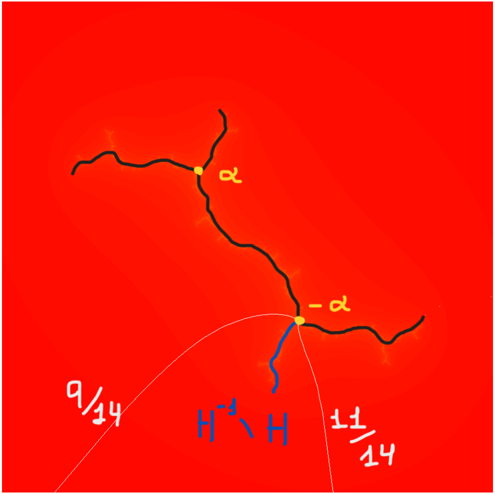


Figure: $H_{1/4}^{-1}$ and angle sector $[9/14, 11/14]$.

The Forbidden region

Proposition

An external ray $\mathcal{R}(t)$ lands on the Hubbard tree H_θ if and only if the orbit of t under doubling never enters the forbidden region.

The forbidden region for $\theta = 1/4$

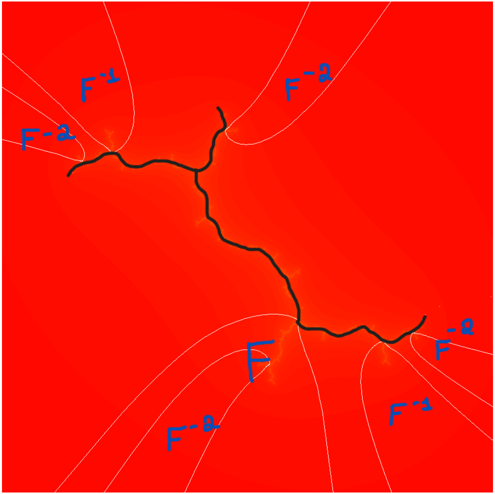


Figure: Forbidden region F and preimages $f^{-1}(F)$ and $f^{-2}(F)$.

From forbidden regions to automata

Set of angles landing on H_θ: $S^1 \setminus \bigcup_{n \geq 0} \sigma^{-n}(F)$, where $\sigma(t) = 2t$.

Given $t \in S^1$, to check if its ray lands on H_θ, need to know its orbit under doubling.

From forbidden regions to automata

Set of angles landing on H_θ: $S^1 \setminus \bigcup_{n\geq 0} \sigma^{-n}(F)$, where $\sigma(t) = 2t$.

Given $t \in S^1$, to check if its ray lands on H_θ, need to know its orbit under doubling.

Another point of view: set up a Markov partition on S^1 from the forbidden region.

A Markov partition for $\theta = 1/4$

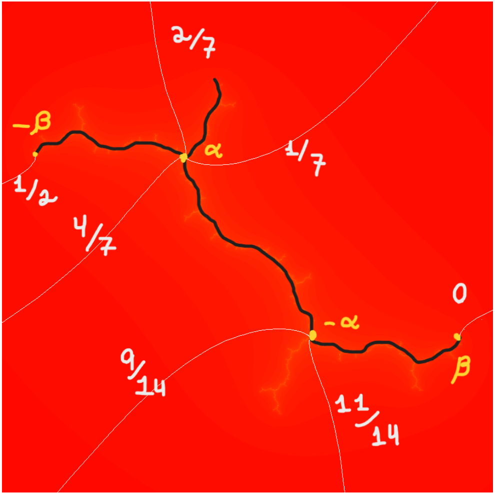


Figure: Forward orbit of endpoints of forbidden region.

A Markov partition for $\theta = 1/4$

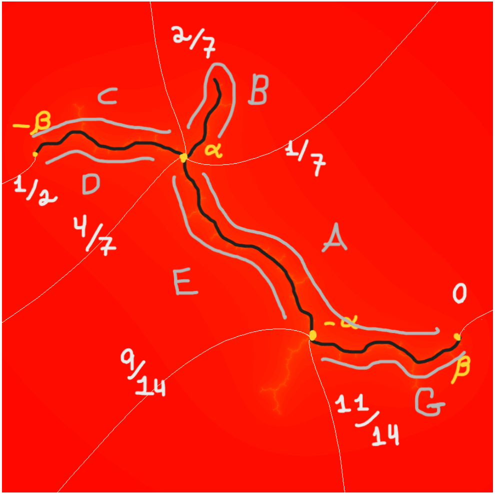
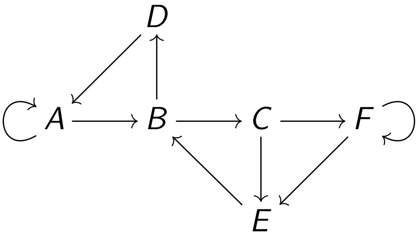


Figure: Labeling non-forbidden arcs.

The finite state automaton

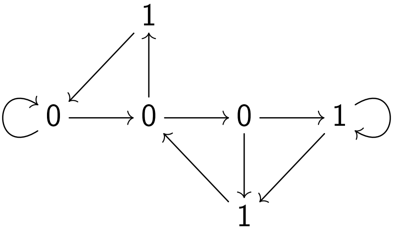
Transition graph/**finite state automaton**:



Example of $\theta = 1/4$ worked out by Milnor.

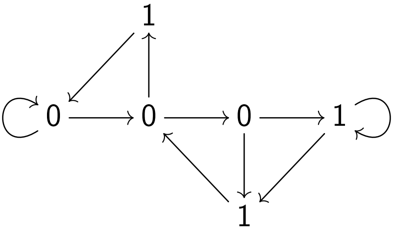
The finite state automaton

Each symbol S has an associated bit value $b(S)$, whether it above the spine $[\beta, -\beta]$ or not:



The finite state automaton

Each symbol S has an associated bit value $b(S)$, whether it above the spine $[\beta, -\beta]$ or not:



Proposition

An angle $t = .b_1 b_2 b_3 \ldots$ in binary has external ray $\mathcal{R}(t)$ landing on H_θ if and only if the infinite string $b_1 b_2 b_3 \ldots$ is recognized by this automaton.

Examples

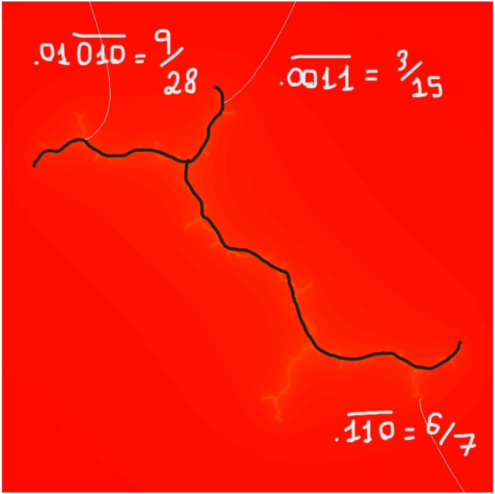


Figure: 3/15 and 9/28 land on $H_{1/4}$, but 6/7 doesn't.

Finding the forbidden region

How to actually find the forbidden region?

Finding the forbidden region

How to actually find the forbidden region?

- Attaching points are strictly preperiodic
- Their periodic orbits have branch number $q \geq 3$
- Correspond to satellite bifurcations along the "path to" c_θ
- Find the **angled internal address** of c_θ

Finding the forbidden region

How to actually find the forbidden region?

- Attaching points are strictly preperiodic
- Their periodic orbits have branch number $q \geq 3$
- Correspond to satellite bifurcations along the "path to" c_θ
- Find the **angled internal address** of c_θ

Idea: moving from the main cardioid to c_θ, record the "landmark" hyperbolic components along the way.

Internal addresses

Recursive definition: for $c \in M$,

- $S_0 = 1$, $(\theta_0, \theta_0') = (0, 1)$

Internal addresses

Recursive definition: for $c \in M$,

- $S_0 = 1$, $(\theta_0, \theta'_0) = (0, 1)$
- Given S_k and (θ_k, θ'_k), let $\mathcal{R}(\theta_{k+1}, \theta'_{k+1})$ be the parameter ray pair *of lowest period* separating c_θ from $\mathcal{R}(\theta_k, \theta'_k)$
- S_{k+1} is the period of the component W_{k+1} that $\mathcal{R}(\theta_{k+1}, \theta'_{k+1})$ lands at the root of

Internal addresses

Recursive definition: for $c \in M$,

- $S_0 = 1$, $(\theta_0, \theta'_0) = (0, 1)$
- Given S_k and (θ_k, θ'_k), let $\mathcal{R}(\theta_{k+1}, \theta'_{k+1})$ be the parameter ray pair *of lowest period* separating c_θ from $\mathcal{R}(\theta_k, \theta'_k)$
- S_{k+1} is the period of the component W_{k+1} that $\mathcal{R}(\theta_{k+1}, \theta'_{k+1})$ lands at the root of

Creates sequence

$$S_0 \longrightarrow S_1 \longrightarrow S_2 \longrightarrow \cdots \longrightarrow S_k \longrightarrow \cdots$$

Internal addresses

Recursive definition: for $c \in M$,

- $S_0 = 1$, $(\theta_0, \theta'_0) = (0, 1)$
- Given S_k and (θ_k, θ'_k), let $\mathcal{R}(\theta_{k+1}, \theta'_{k+1})$ be the parameter ray pair *of lowest period* separating c_θ from $\mathcal{R}(\theta_k, \theta'_k)$
- S_{k+1} is the period of the component W_{k+1} that $\mathcal{R}(\theta_{k+1}, \theta'_{k+1})$ lands at the root of

Creates sequence

$$S_0 \longrightarrow S_1 \longrightarrow S_2 \longrightarrow \cdots \longrightarrow S_k \longrightarrow \cdots$$

Remark: Equivalent to kneading sequence of θ, if $c = c_\theta$.

Internal address of $c_{1/4}$

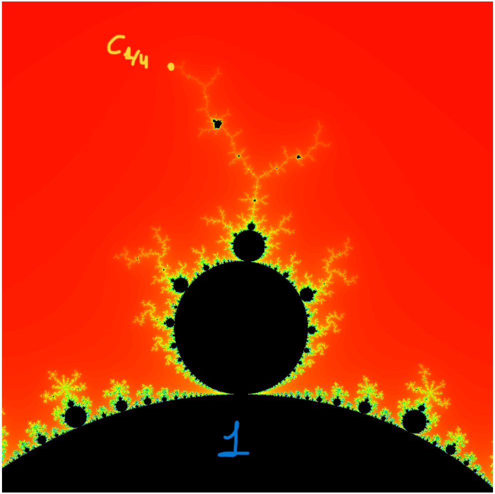


Figure: Internal address $1 \to 3 \to 4 \to 5 \to \cdots$

Internal address of $c_{1/4}$

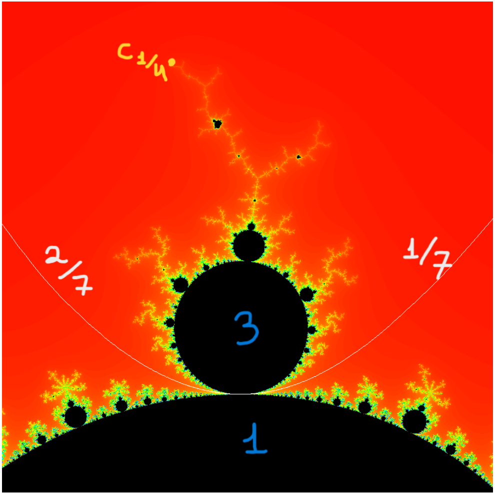


Figure: Internal address $1 \to 3 \to 4 \to 5 \to \cdots$

Internal address of $c_{1/4}$

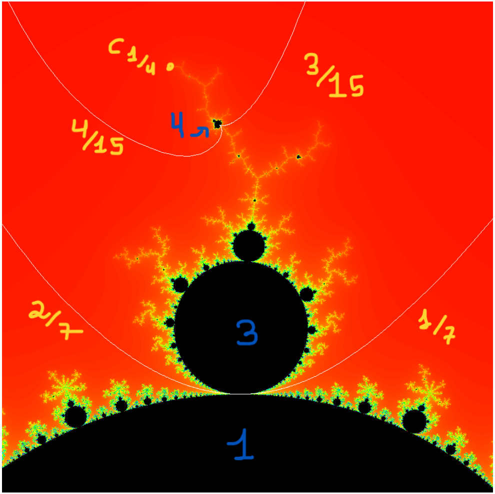


Figure: Internal address $1 \to 3 \to 4 \to 5 \to \cdots$

Angled internal address

If c is in the p_k/q_k-sublimb of W_k, we add this information to the internal address:

$$(S_0)_{p_0/q_0} \longrightarrow (S_1)_{p_1/q_1} \longrightarrow \cdots \longrightarrow (S_k)_{p_k/q_k} \longrightarrow \cdots$$

Angled internal address

If c is in the p_k/q_k-sublimb of W_k, we add this information to the internal address:

$$(S_0)_{p_0/q_0} \longrightarrow (S_1)_{p_1/q_1} \longrightarrow \cdots \longrightarrow (S_k)_{p_k/q_k} \longrightarrow \cdots$$

Tells you which direction to follow to get to c.

Angled internal address

If c is in the p_k/q_k-sublimb of W_k, we add this information to the internal address:

$$(S_0)_{p_0/q_0} \longrightarrow (S_1)_{p_1/q_1} \longrightarrow \cdots \longrightarrow (S_k)_{p_k/q_k} \longrightarrow \cdots$$

Tells you which direction to follow to get to c.

The angled internal address uniquely describes the **combinatorial class** of the parameter c!

Angled internal address

If c is in the p_k/q_k-sublimb of W_k, we add this information to the internal address:

$$(S_0)_{p_0/q_0} \longrightarrow (S_1)_{p_1/q_1} \longrightarrow \cdots \longrightarrow (S_k)_{p_k/q_k} \longrightarrow \cdots$$

Tells you which direction to follow to get to c.

The angled internal address uniquely describes the **combinatorial class** of the parameter c!

Angled internal address is computable from θ; knowing it, can find all periodic branch points of H_θ.

Angled internal address

If c is in the p_k/q_k-sublimb of W_k, we add this information to the internal address:

$$(S_0)_{p_0/q_0} \longrightarrow (S_1)_{p_1/q_1} \longrightarrow \cdots \longrightarrow (S_k)_{p_k/q_k} \longrightarrow \cdots$$

Tells you which direction to follow to get to c.

The angled internal address uniquely describes the **combinatorial class** of the parameter c!

Angled internal address is computable from θ; knowing it, can find all periodic branch points of H_θ.

$\theta = 1/4$:

$$(1)_{1/3} \longrightarrow (3)_{1/2} \longrightarrow (4)_{1/2} \longrightarrow \cdots$$

From angled internal address to forbidden region

Knowing the angled internal address:

$$(S_0)_{p_0/q_0} \longrightarrow (S_1)_{p_1/q_1} \longrightarrow \cdots \longrightarrow (S_k)_{p_k/q_k} \longrightarrow \cdots$$

Possible to compute the rays landing at periodic branch points of K_c.*

From angled internal address to forbidden region

Knowing the angled internal address:

$$(S_0)_{p_0/q_0} \longrightarrow (S_1)_{p_1/q_1} \longrightarrow \cdots \longrightarrow (S_k)_{p_k/q_k} \longrightarrow \cdots$$

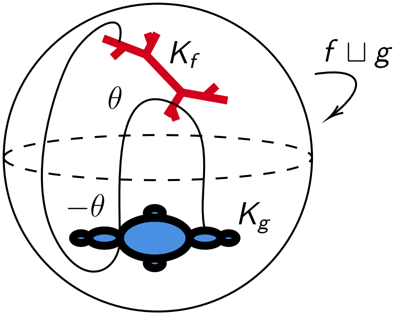
Possible to compute the rays landing at periodic branch points of K_c.*

By looking at iterated preimages, can compute the forbidden region.

Applications to matings

Mating of two polynomials f and g:

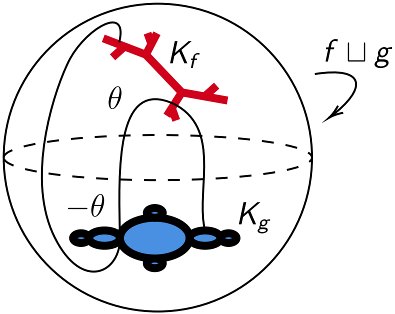
Attach circles at infinity, glue along the circles to form a 2-sphere



Applications to matings

Mating of two polynomials f and g:

Attach circles at infinity, glue along the circles to form a 2-sphere



Collapse external rays to points! Quotient of $K_f \sqcup K_g$.

Ray connections between Hubbard trees

What are the angles t such that $\mathcal{R}(t)$ lands on H_{θ_1}, and $\mathcal{R}(-t)$ lands on H_{θ_2}?

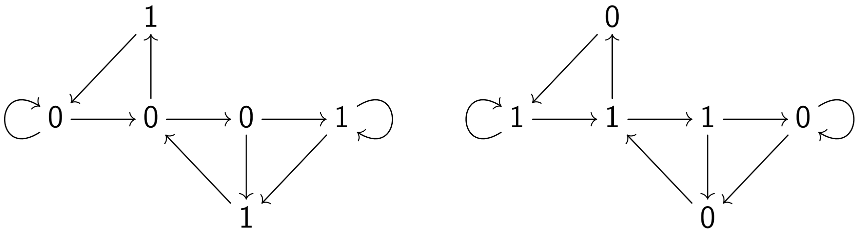
Ray connections between Hubbard trees

What are the angles t such that $\mathcal{R}(t)$ lands on H_{θ_1}, and $\mathcal{R}(-t)$ lands on H_{θ_2}?

Mate the automata:

- Flip the bits on A_{θ_2};
- Find the sequences admissible for both, simultaneously.

Example: $\theta_1 = \theta_2 = 1/4$



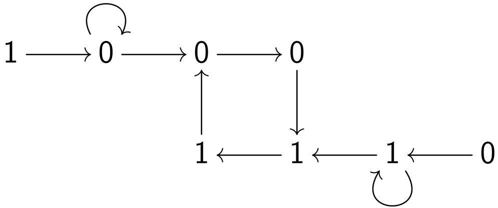
Example: $\theta_1 = \theta_2 = 1/4$

```
In [117]: hta.mating_dyadics(Fraction(1,4), Fraction(1,4))
Out[117]:
[[1 0 0 0 1 0 0 0]
 [1 0 0 0 0 1 0 0]
 [0 0 0 0 0 0 0 0]
 [0 1 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 1]
 [0 0 1 0 0 0 1 0]
 [0 0 0 1 0 0 1 0]]
```

Figure: Adjacency matrix of the mating automaton.

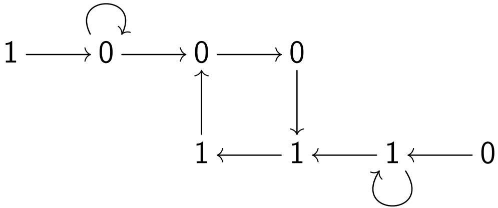
Example: $\theta_1 = \theta_2 = 1/4$

Mating automaton with bits:



Example: $\theta_1 = \theta_2 = 1/4$

Mating automaton with bits:



Only periodic ray connections are $t = 0$ and

$$t \in \left\{ \frac{3}{15}, \frac{6}{15}, \frac{12}{15}, \frac{9}{15} \right\}.$$

All others are preperiodic.

Example: $\theta_1 = \theta_2 = 1/4$

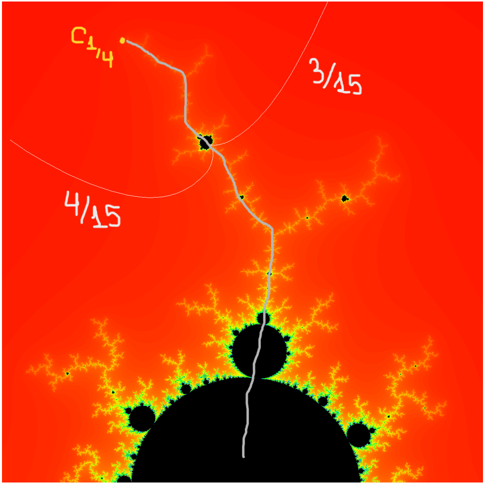


Figure: In the "path" to $c_{1/4}$, where the ray connection is "picked up".

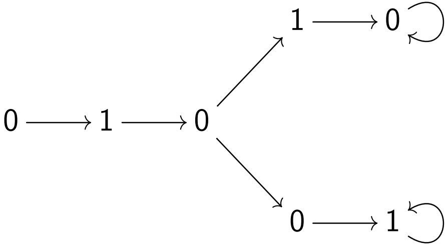
Example: $\theta_1 = 1/4$, $\theta_2 = 11/32$

```
In [19]: mating_dyadics(Fraction(1,4), Fraction(11,32))
Out[19]:
[[1 0 0 0 1 0 0]
 [0 0 0 0 0 1 0]
 [0 1 0 0 0 0 0]
 [0 0 0 0 0 0 0]
 [0 1 0 0 0 0 0]
 [0 0 0 1 0 0 0]
 [0 0 1 0 0 0 1]]
```

Figure: Adjacency matrix of the mating automaton.

Example: $\theta_1 = 1/4$, $\theta_2 = 11/32$

Mating automaton with bits:



Only ray connections for $\mathcal{R}(t)$ landing on $H_{1/4}$ are:

$$\left\{ \frac{5}{16}, \frac{5}{8}, \frac{1}{4}, \frac{1}{2}, \frac{0}{1} \right\}$$

Example: $\theta_1 = 1/4$, $\theta_2 = 11/32$

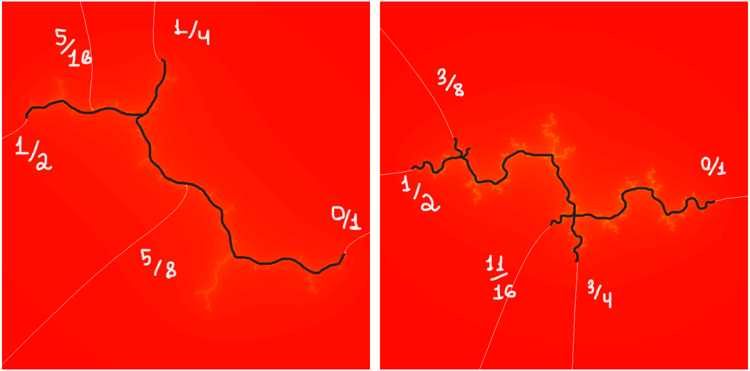


Figure: Ray connections between $H_{1/4}$ and $H_{11/32}$.

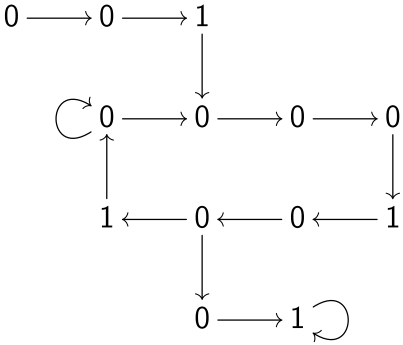
Example: $\theta_1 = 1/8$, $\theta_2 = 23/32$

```
In [28]: mating_dyadics(Fraction(1,8), Fraction(23,32))
Out[28]:
[[1 0 0 0 0 0 0 0 0 1 0 0 0]
 [0 0 0 0 0 0 0 0 0 0 1 0 0]
 [1 0 0 0 0 0 0 0 0 0 0 1 0]
 [0 1 0 0 0 0 0 0 0 0 0 0 0]
 [0 0 1 0 0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0 0 0 0 0]
 [0 0 0 1 0 0 0 0 0 0 0 0 0]
 [0 0 0 0 1 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 1 0 0 0 0 0 0 0]
 [0 0 0 1 0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 1 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 1 0 0 0 0]
 [0 0 0 0 0 0 1 0 0 0 0 0 1]]
```

Figure: Adjacency matrix of the mating automaton.

Example: $\theta_1 = 1/8$, $\theta_2 = 23/32$

Ray connections landing on $H_{1/8}$:



Entropy is ≈ 0.2086.

Questions and goals

Some questions:

- When are there no ray connections, apart from $1/2$ and $0/1$?
- When is the entropy positive? When is it zero?
- How are the Hubbard trees embedded in the geometric mating?

Questions and goals

Some questions:

- When are there no ray connections, apart from $1/2$ and $0/1$?
- When is the entropy positive? When is it zero?
- How are the Hubbard trees embedded in the geometric mating?

Some goals:

- Extend algorithm to non-dyadic rationals
- Compute automata encoding ray connections of length ℓ

Entropy of mating with $1/4$

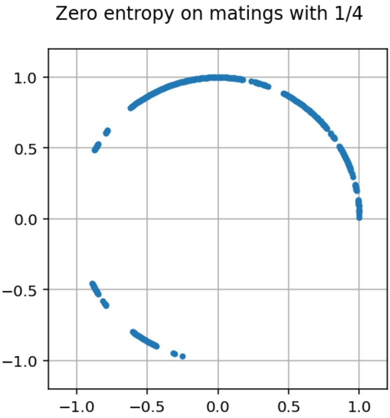


Figure: Angles $a/1024$ having zero entropy on mating with $1/4$

Thank you!

Thank you!

Python code available at
github.com/evsmath/Hubbard-tree-automata

References I

[DH85] A. Douady and J. Hubbard. "Étude Dynamique des Polynômes Complexes I et II". In: *Publ. Math. Orsay* (1984-1985).

[Dou86] A. Douady. "Algorithms for computing angles in the Mandelbrot set". In: *Chaotic Dynamics and Fractals* (1986), pp. 155–168.

[DS20] D. Dudko and D. Schleicher. "Core Entropy of Quadratic Polynomials". In: *Arnold Math J.* 6 (2020), pp. 333–385.

[Jun14] W. Jung. "Core entropy and biaccessibility of quadratic polynomials". In: *arXiv* (2014).

[Jun17] W. Jung. "Quadratic matings and ray connections". In: *arXiv* (2017).

References II

[Mil04] J. Milnor. "Pasting Together Julia Sets: A Worked Out Example of Mating". In: *Experimental Mathematics* 13 (2004), pp. 55–92.

[Sch17] D. Schleicher. "Internal Addresses of the Mandelbrot Set and Galois Groups of Polynomials". In: *Arnold Math J.* 3 (2017), pp. 1–35.