Milnor's Dynamics in One Complex Variable: Problems and Thoughts

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The following consists of solutions (mostly complete, a few partial) to problems in Milnor's *Dynamics in One Complex Variable* ([2]). I initially started with the second edition, but for later chapters I also used the third. I also include personal thoughts and clarifications for some arguments that I thought were vague or could use more explaining.

Contents

1 Simply Connected Surfaces

Problem (1-d. Conjugacy classes in $\mathcal{G}(\mathbb{H})$)**.** Show that every automorphism of $\mathbb{H}$ without fixed points is conjugate to a unique transformation of the form $w \mapsto w+1$ or $w \mapsto w-1$ or $w \mapsto \lambda w$ with $\lambda > 1$; and show that the conjugacy class of an automorphism g with fixed point $w_0 \in \mathbb{H}$ is uniquely determined by the derivative $\lambda = g'(w_0)$, where $|\lambda| = g'(w_0)$.

Proof. If g has a fixed point w_0, then by equivalently considering $g : \mathbb{D} \to \mathbb{D}$ and conjugating g by

$$z \mapsto \frac{z-a}{1-\bar{a}z},$$

for a the image of w_0 in $\mathbb{D}$, we may assume g fixes the origin 0. Hence by Schwarz's lemma g is a rotation, where $g'(0) = e^{i\theta} = \lambda$. Since conjugation by holomorphic maps must preserve derivatives at fixed points, this λ is uniquely determined by the conjugacy class.

Now suppose g has no fixed points in $\mathbb{H}$. If g, as a Möbius transformation on $\hat{\mathbb{C}}$, has a unique fixed point on $\hat{\mathbb{R}}$, we may conjugate by a projective linear map on $\hat{\mathbb{R}}$, which preserves $\mathbb{H}$, to assume the fixed point is at ∞. But then g is of the form $az + b$, and since it has no fixed points on $\mathbb{C}$, g is of the form $g(z) = z + b$. Since it must preserve $\mathbb{H}$, we must have $b \in \mathbb{R}$. Then, by conjugating by $h(z) = \frac{1}{|b|}z$, we obtain $z \pm 1$, depending on whether b is positive or negative. The uniqueness of the conjugacy class is easily seen, as $z \mapsto z + 1$ cannot be conjugate to $z \mapsto z - 1$: if a fractional linear transformation h conjugated the maps, an easy calculation gives us that its determinant would have to be -1, whereby we may take it to be an element of $\mathrm{PSL}(2,\mathbb{R})/\{\pm I\}$.

If g had two fixed points on $\hat{\mathbb{R}}$, by an appropriate conjugation we may assume they are 0 and ∞, and a similar analysis would give us that $g(z) = \lambda z$, for $\lambda > 0$, in order to preserve $\mathbb{H}$. We may assume $\lambda \neq 1$ so that it is not the identity. It is also easy to see that we have a conjugation between $z \mapsto \lambda z$ and $z \mapsto \lambda^{-1} z$, and no other maps in this class.

□

Problem (1-h. Convergence to zero)**.** If a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ fixes the origin, and is not a rotation, prove that the sucessive images $f^n(z)$ converge to zero for all $z \in \mathbb{D}$, and prove that this convergence is uniform on all compact subsets of $\mathbb{D}$.

Proof. Since f is not a rotation, by Schwarz's lemma, we have that $|f'(z)| < 1$ for all $z \in \mathbb{D}$, and $|f(z)| < |z|$ for all $z \neq 0$. Hence, in any compact subset $K \subset \mathbb{D}$, by continuity of f, there exists some $\lambda = \lambda_K < 1$ such that $|f(z)| \leq \lambda |z|$ for $z \in K$. Hence $|f^n(z)| \leq \lambda^n |z| \leq \lambda^n$ for $z \in K$, so that $f^n \to 0$ uniformly in K. □

2 Universal Coverings and the Poincaré Metric

Problem (2-b. Lifting to the Universal Covering)**.** If $S \cong \mathbb{D}/\Gamma$ and $S' \cong \mathbb{D}/\Gamma'$ are hyperbolic surfaces, show that any holomorphic map $f : S \to S'$ lifts to a holomorphic map $F : \mathbb{D} \to \mathbb{D}$, unique up to composition with an element of Γ'. Show that f induces a group homomorphism from Γ to Γ' satisfying the identity

$$F \circ \gamma = \gamma' \circ F,$$

for every $\gamma \in \Gamma$. Show that f is a covering map if and only if F is a conformal automorphism.

Proof. Suppose for now that f is only continuous. If $p : \mathbb{D} \to S$ and $p' : \mathbb{D} \to S'$ are the convering projections, then $f \circ p : \mathbb{D} \to S'$ is a continuous map. As $\mathbb{D}$ is simply connected, fixing $z_0 \in \mathbb{D}$ and $w \in p'^{-1}(f(p(z_0)))$, by the theory of covering spaces there exists a unique continuous map $F : \mathbb{D} \to \mathbb{D}$ that lifts $p \circ f$ to $\mathbb{D}$ and such that $F(z_0) = w$. Since any other element of the fiber $w \in p'^{-1}(f(p(z_0))$ is given by $w' = \gamma' w$ for a unique $\gamma' \in \Gamma'$, and $\gamma' \circ F$ satisfies the condition, we indeed have uniqueness of the lift up to post-composition with elements of Γ'.

If $\gamma \in \Gamma$, then $F \circ \gamma$ is also a lift of γ, and by the above, there exists a unique $\gamma' \in \Gamma'$ such that $F \circ \gamma = \gamma' \circ F$. This resulting map $\mathcal{F} : \Gamma \to \Gamma'$ is a group homomorphism, as can be see from

$$F \circ \gamma \circ \eta = \gamma' \circ F \circ \eta = \gamma' \circ \eta' \circ F.$$

We note the following. If F and $G = \sigma \circ F$ are two lifts of f, where $\sigma \in \Gamma'$, then the group homomorphisms $\mathcal{F}, \mathcal{G}$ are related by the following:

$$\begin{gathered} F \circ \gamma = \mathcal{F}(\gamma) \circ F, \quad G \circ \gamma = \mathcal{G}(\gamma) \circ G \\ \implies \sigma \circ F \circ \gamma = \mathcal{G}(\gamma) \circ \sigma \circ F \\ \implies \sigma \circ \mathcal{F}(\gamma) \circ F = \mathcal{G}(\gamma) \circ \sigma \circ F, \end{gathered}$$

so that, by uniqueness of the lifts up to post-composition,

$$\mathcal{G}(\gamma) = \sigma \mathcal{F}(\gamma) \sigma^{-1}.$$

Hence the group homomorphism from the fundamental groups of S and S' is not uniquely given by just f, but any two such group homomorphisms are related by post-conjugation in Γ', corresponding to distinct choices of lifts of f.

A case of note is when f is a homeomorphism; then, by functioriality, it is easily deduced that the lifts F will be homeomorphisms, and the group homomorphism $\mathcal{F} : \Gamma \to \Gamma'$ will be an isomorphism. Again, this is not uniquely given by f, but only up to post-conjugation.

From the theory of covering spaces, F is a homeomorphism if and only if f is a covering map, and if F is holomorphic, then it will be a conformal automorphism of $\mathbb{D}$. □

Problem (2-f. Infinite band, cylinder, and annulus)**.** Define the *infinite band* $B \subset \mathbb{C}$ of height π to be the set $x + iy \in \mathbb{C}\colon |y| < \pi/2\}$. Show that the exponential map carries B isomorphically onto the right-half plane. Show that the Poincaré metric on B takes the form $ds = |dz|/\cos y$.

Show that the real axis is a geodesic whose Poincaré arclength coincides with its usual Euclidean arclength, and show that each translation $z \mapsto z + c$ is a hyperbolic automorphism of B having the real axis as its unique invariant geodesic. For any $c > 0$, form the quotient cylinder $S_c = B/(c\mathbb{Z})$ by identifying each $z \in B$ with $z + c$. By definition, the *modulus* $\mod (S_c)$ of the resulting cylinder is the ration π/c of height to circumference. Show that this cylinder, with the Poincaré metric, has a unique simple closed geodesic, with length $c = \pi/ \mod (S_c)$.

Show that S_c is conformally isomorphic to the annulus $\mathbb{A}_r = \{z \in \mathbb{C}\colon 1 < |z| < r\}$ where $\log r = 2\pi^2/c$. Conclude that $\mod (\mathbb{A}_r) = \log r/2\pi$ is a conformal invariant.

Proof. We see that

$$\exp(x+iy) = e^x(\cos y + i\sin y) = e^x\cos y + ie^x \sin y,$$

so that, for every $u+iv$ with $u>0$, we have unique x and y with $|y|<\pi/2$ such that $x+iy$ maps to $u+iv$, considering the polar form of $u+iv$. The resulting map will be bijective and holomorphic, so that $f : z \mapsto i\exp(z)$ is a conformal isomorphism from B to $\mathbb{H}$. The pullback of the Poincaré metric is

$$f^*\left(\frac{1}{v^2}|dw|^2\right) = \frac{1}{|ie^x\cos y|}|ie^z||dz| = \frac{1}{\cos y}|dz|.$$

The conformal automorphism, being an isometry, maps the real axis onto the positive imaginary axis on $\mathbb{H}$, and since it is a geodesic, the real axis will also be. Given that for $y=0$, $\cos y = 1$, we have that the inclusion $\mathbb{R}\subset B$ pulls back the metric on B to the standard metric $|dx|$ on $\mathbb{R}$. The translations $\tau_c(z) = z+c$, for $c\in\mathbb{R}$, are conformal isometries of B, since they are conformal automorphisms which pullback the metric on B to itself.

The real axis $\mathbb{R}$ must be the unique geodesic in B invariant under τ_c, as τ_c maps to the dilation $w\mapsto e^c w$ in $\mathbb{H}$. We know the geodesics of $\mathbb{H}$ are vertical lines or the semicircunferences orthogonal to the real axis, and the only one that is invariant by a positive scaling is the imaginary axis.

Under the projection $p : B \to S_c$, the real axis becomes a simple closed geodesic in S_c. Suppose γ were another closed geodesic in S_c, and let $\tilde{\gamma}$ denote one of its lifts to B. We see that all lifts of γ are in fact given by $\tau_c^n \circ \tilde{\gamma}$, for $n\in\mathbb{N}$.

If $\gamma : \mathbb{R}\to S_c$ is a closed geodesic, there must exist $t_0, t_1 \in \mathbb{R}$ distinct such that $\gamma(t_0) = \gamma(t_1)$, $\gamma'(t_0) = \gamma'(t_1)$. By uniqueness of the geodesic equation, we have that $\gamma(t) = \gamma(t+(t_1-t_0))$ for all $t\in\mathbb{R}$, so that γ is periodic. Supposing that γ is not constant, it has a minimal period δ, so that $\gamma(t+\delta) = \gamma(t)$ for all t, and is injective in $[t, t+\delta)$.

We then have that $\tilde{\gamma}(\delta) = \tilde{\gamma}(0) + nc$ for some $n\in\mathbb{N}$. If $n=0$, then $\tilde{\gamma}$ forms a closed C^1 loop in $B\subset\mathbb{C}$, hence must at some point have strictly vertical tangent vector. But by uniqueness of the geodesic equation, $\tilde{\gamma}$ must be a vertical line, a contradiction. Hence $n\neq 0$. Moreover, since $\tilde{\gamma}(t+\delta)-nc$ is also a lift of γ having the same starting point as $\tilde{\gamma}$ at $t=0$, we have more generally that $\tilde{\gamma}(t+\delta) = \tilde{\gamma}(t)+nc$. This shows that $\tilde{\gamma}$ stays a bounded hyperbolic distance of the real axis $\mathbb{R}$ as it goes to infinity.

In $\mathbb{H}$, this means that the geodesic η corresponding to $\tilde{\gamma}$ satisfies $\eta(t+\delta) = e^{cn}\eta(t)$ for all t. This property cannot be satisfied either by a vertical line that is not the imaginary axis, nor by a circumference orthogonal to $\partial\mathbb{H}$; hence we obtain a contradiction. (This in fact proves that the projection of $\mathbb{R}$ is the unique closed geodesic in S_c, not just simple.)

The length of this unique simple closed geodesic is the same as the euclidean length of the segment on which the projection to S_c is injective, hence c. Finally, consider exponential map

$$\exp : \{x + iy \in \mathbb{C} \colon 0 < x < \log r\} \to A_r,$$

which is a covering map on this strip S. This strip is naturally conformally isomorphic to B by the map $h : B \to S$ given by

$$h(z) = (\log r)(\frac{i}{\pi} z + 1).$$

We note that $\exp \circ h$ is injective when $0 \leq x < c$, and we get a quotient map $S_c \to A_r$ which is going to be a conformal isomorphism. □

Problem (2-g. Abelian Fundamental Groups)**.** Show that every hyperbolic surface with abelian fundamental group is conformally isomorphic either to the disk $\mathbb{D}$, or to the punctured sphere $\mathbb{D} \setminus \{0\}$, or to an annulus A_r for some uniquely identified $r > 1$. Show that this annulus has a unique simple closed geodesic, which has length $l = 2\pi^2/\log r$. On the other hand, show that the punctured disk $\mathbb{D} \setminus \{0\}$ has no closed geodesic. Show that the conformal automorphism group $\mathcal{G}(\mathbb{D} \setminus \{0\})$ of a punctured disk is isomorphic to the circle group SO(2), while the conformal automorphism group of an annulus is isomorphic to the non-abelian group O(2). What is the automorphism group for $\mathbb{C} \setminus \{0\}$?

Proof. Suppose $S \cong \mathbb{D}/\Gamma$, where Γ is abelian, discrete and has no fixed points. If $\Gamma = \{e\}$, then $S \cong \mathbb{D}$, so we assume Γ is not the trivial subgroup. Since it has no fixed points, it contains no elliptic elements. We know that two non-identity elements in $\mathcal{G}(\mathbb{D})$ commute if and only if they have the same fixed point set in $\overline{\mathbb{D}}$. So either this fixed point set shared by all elements of Γ is a unique point in $\partial\mathbb{D}$, where Γ consists only of parabolic elements, or two points in $\partial\mathbb{D}$, so that Γ consists only of hyperbolic elements.

In the first case, we may assume that the fixed point is $\infty \in \partial\mathbb{H}$, so that Γ is a discrete subgroup of translations of $\mathbb{R}$. But then it must be isomorphic to $\mathbb{Z}$ and be generated by a single translation. In the second case, we may assume the fixed points are 0 and ∞ in $\partial\mathbb{H}$. Hence the hyperbolic elements must preserve the geodesic from 0 to ∞, the positive imaginary axis, acting by dilations. In the band model B, they act by translations on the real axis, so similarly $\Gamma \cong \mathbb{Z}$ and they are generated by a single translation.

If γ is a generator for Γ, it must either be a parabolic or hyperbolic element of $\mathcal{G}(\mathbb{D})$. Conjugate discrete subgroups of $\mathcal{G}(\mathbb{D})$ without fixed points give rise to conformally isomorphic Riemann surfaces via the mapping

$$\begin{array}{ccc} \mathbb{D}/\Gamma & \to & \mathbb{D}/(\sigma\Gamma\sigma^{-1}) \\ [a] & \to & [\sigma a] \end{array}$$

so we only care about the conjugacy class of γ. If γ is parabolic, then by taking γ^{-1} also a generator of Γ, we may assume it is conjugate to $w \mapsto w+1$ in $\mathbb{H}$. The map $f : \mathbb{H} \to \mathbb{D} \setminus \{0\}$ given by $f(z) = \exp(2\pi i z)$ is a universal covering map of $\mathbb{D} \setminus \{0\}$, which descends to an isomorphism via the identification by translation $z \sim z+1$. Hence $S \cong \mathbb{D} \setminus \{0\}$.

If γ is hyperbolic, it is a translation by c on the band model, where we have proved that $S \cong A_r$, for the specified length. We have also already shown the uniqueness of the simple closed geodesic, as this is a property of A_r; it is the projection of the geodesic that the hyperbolic elements of Γ preserve.

Suppose γ is a closed geodesic in $\mathbb{D} \setminus \{0\} \cong \mathbb{H}/\mathbb{Z}$. If $\tilde{\gamma}$ is a lift of γ to $\mathbb{H}$, by similar arguments as before, there exists some $n \in \mathbb{N}$ and some $\delta > 0$ such that, for all $t \in \mathbb{R}$, $\tilde{\gamma}(t+\delta) = \tilde{\gamma}(t) + n$. But no geodesic in $\mathbb{H}$ can satisfy this property for all t.

It is easy to see that $\mathrm{SO}(2) \subseteq \mathcal{G}(\mathbb{D} \setminus \{0\})$, acting by rotational symmetries around 0. These correspond to translations $w \mapsto w+t$ in $\mathbb{H}/\mathbb{Z}$. Suppose $\sigma \in \mathcal{G}(\mathbb{D} \setminus \{0\})$. By problem 2-b, σ lifts to a conformal automorphism $\tilde{\sigma}$ of $\mathbb{H}$, and a group isomorphism $\mathcal{F} : \mathbb{Z} \to \mathbb{Z}$ such that

$$\tilde{\sigma}(w+1) = \tilde{\sigma}(w) \pm 1,$$

depending on which generator the translation $w \mapsto w+1$ which generates $\mathbb{Z}$ gets mapped to. But it must be $+1$, since $w \mapsto w+1$ and $w \mapsto w-1$ are not conjugate in $\mathbb{H}$. Hence

$$\tilde{\sigma}(w+1) = \tilde{\sigma}(w) + 1,$$

commuting with translations. Then $\tilde{\sigma} \in \mathrm{PSL}(2,\mathbb{R})/\{\pm I\}$ must itself be a translation, which indeed descends to a conformal automorphism of $\mathbb{D} \setminus \{0\}$ of the form given by $\mathrm{SO}(2)$.

Now consider $G = \mathcal{G}(A_r)$, $r > 1$. Any $\sigma \in G$ lifts to a conformal automorphism $\widetilde{\sigma} : B \to B$ such that

$$\widetilde{\sigma}(z+l) = \widetilde{\sigma}(z) \pm l,$$

or equivalently, an automorphism of $\mathbb{H}$ such that

$$\widetilde{\sigma}(\lambda w) = \lambda^{\pm 1}\widetilde{\sigma}(w).$$

This is a different case because λz and $\lambda^{-1} z$ are indeed conjugate in $\mathcal{G}(\mathbb{H})$. The resulting characterization of $\mathcal{G}(A_r)$ will be $\mathrm{O}(2)$, with an element that "flips" the annulus inside out, and interchanges the boundaries, while simultaneously preserving orientation.

(I don't want to do the case $\mathcal{G}(\mathbb{C} \setminus \{0\})$.) $\square$

Problem (2-k. No non-trivial holomorphic attractors)**.** If $K \subset S$ is compact with $f(K) = K$, and if f maps some connected hyperbolic neighborhood U of K into a proper subset of itself, show that f must be stricly contracting on K with respect to the metric d_U, and hence that K must consist of a single point.

Proof. Considering U as a hyperbolic Riemann surface, since $f : U \to U$ is holomorphic and non-surjective, we have that, by Pick's theorem, there exists $\lambda < 1$ such that, for all $p, q \in K$, $d_U(f(p), f(q)) \leq \lambda d_U(p, q)$. Hence f_K is a uniform contraction, and by the Banach fixed point theorem, all points must convergence to a fixed point under iteration. As $f^n(K) = K$, K must consist of this single point. $\square$

Problem (2-m. The Picard theorem near infinity.)**.** Prove the following statement in two steps, as indicated.

Theorem 2.1 (Picard)**.** *Any holomorphic map $f : \mathbb{D} \setminus \{0\} \to \hat{\mathbb{C}} \setminus \{a, b, c\}$ to the triply punctured sphere extends to a holomorphic map from $\mathbb{D}$ to $\hat{\mathbb{C}}$.*

(1) Prove the statement for the special case where $f(z)$ converges to a as $z \to 0$.

(2) On the other hand, suppose that $f(z)$ does not converge to a, b or c as $z \to 0$. Show that there must exist some point $p \in \hat{\mathbb{C}} \setminus \{a, b, c\}$ which is an accumulation point of images $f(z)$ as $z \to 0$. Using the Poincaré metric as described in example 2.8, show that the image of a small circle $|z| = r$ lies in a small neighborhood of p. Conclude that f restricted to this circle lifts to the universal covering space of $\hat{\mathbb{C}} \setminus \{a, b, c\}$, and hence that f on the entire punctured disk lifts to this covering space, and use this to complete the proof.

(3) Now apply this result for a circle centered at ∞ to prove the following.

Theorem 2.2 (Strong Picard Theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic but not a polynomial, then for every neighborhood $\mathbb{C} \setminus \overline{\mathbb{D}_r}$ of infinity the image $f(\mathbb{C} \setminus \overline{\mathbb{D}_r})$ omits at most a single point of $\mathbb{C}$. In fact, f takes on every value in $\mathbb{C}$, with at most one exception, infinitely often.*

Proof. (1) If $f(z) \to a$ as $z \to 0$, then a small disk neighborhood of 0 gets mapped into a bounded disk around a. (In fact, by composing f with a Möbius transformation, we may assume $a = 0$.) Hence, by Riemann's removable singularity theorem, we have a holomorphic extension of f by setting $f(z) = a$.

(2) Let $K_n = \overline{f(\mathbb{D}_{1/n} \setminus \{0\})} \subseteq \hat{\mathbb{C}}$, where the K_n form a decreasing sequence of compact connected sets in $\hat{\mathbb{C}}$. We then find that $\bigcap_{n \geq 1} K_n = K$ is compact, connected and non-empty in $\hat{\mathbb{C}}$. By our hypotheses, K cannot be the singleton sets $\{a\}, \{b\}$ and $\{c\}$, so that it must contain some point $p \in \hat{\mathbb{C}} \setminus \{a, b, c\}$, being an accumulation point of some sequence $f(z_n)$ as $z_n \to 0$.

Recall that $\mathbb{D} \setminus \{0\}$ and $\hat{\mathbb{C}} \setminus \{a, b, c\}$ are hyperbolic surfaces, contracting the respective Poincaré distances. As $2\pi/|\log r|$ is the length of the circle $|z| = r$ in $\mathbb{D} \setminus \{0\}$, it gets mapped to a curve γ_r of length $l(r) \leq 2\pi/|\log r| \to 0$ as $r \to 0$. Let $B_\varepsilon(p) \subset \hat{\mathbb{C}} \setminus \{a, b, c\}$ be a neighborhood of p in $\hat{\mathbb{C}} \setminus \{a, b, c\}$, so that for any $r_0 > 0$, we may find a point $\hat{z} \in \mathbb{D} \setminus \{0\}$ such that $|\hat{z}| < r_0$ and $d(\hat{z}, p) < \varepsilon$. Now consider the circle of radius $|\hat{z}|$, which maps to a curve $\gamma_{|\hat{z}|}$ in $\hat{\mathbb{C}} \setminus \{a, b, c\}$ of length $\leq 2\pi/|\log r_0|$. This implies that for any q in the image of this curve, we have

$$d(p, q) \leq d(p, \hat{z}) + \frac{\pi}{|\log r_0|} \leq \varepsilon + \frac{\pi}{|\log r_0|} < 2\varepsilon,$$

if we take r_0 sufficiently small. Hence the image of this circle gets mapped to a neighborhood $B_{2\varepsilon}(p)$ of p.

If ε is sufficiently small, $B_{2\varepsilon}(p)$ is simply connected, so that the image of this curve is nullhomotopic in $\hat{\mathbb{C}} \setminus \{a, b, c\}$. But since this circle generates the fundamental group of $\mathbb{D} \setminus \{0\}$, the map $f : \mathbb{D} \setminus \{0\} \to \hat{\mathbb{C}} \setminus \{a, b, c\}$ is trivial on the fundamental groups. This implies that it lifts to the universal cover $\widetilde{f} : \mathbb{D} \setminus \{0\} \to \mathbb{D}$, and therefore extends to a map $\widetilde{f} : \mathbb{D} \to \mathbb{D}$. This means that by extending $f(0) = \pi(\widetilde{f}(0))$, we get obtain the desired result, as this extension is continuous and therefore holomorphic.

(3) On the hypotheses of the strong Picard theorem, suppose that for some $r > 0$ we have that $f(\mathbb{C} \setminus \overline{\mathbb{D}_r})$ omits two values of $\mathbb{C}$, say a and b. We then get a holomorphic map $f : \mathbb{C} \setminus \overline{\mathbb{D}_r} \to \hat{\mathbb{C}} \setminus \{a, b, \infty\}$, where $\mathbb{C} \setminus \overline{\mathbb{D}_r}$ is conformally isomorphic to a punctured disk. By Picard's theorem, since f does not extend continuously to ∞, otherwise it would be a polynomial, we get that extends continuously by $f(\infty) = c$, a finite

value (possibly equal to a or b). But this implies that f is bounded and entire, hence constant, a contradiction. Therefore $f(\mathbb{C} \setminus \overline{\mathbb{D}_r})$ must omit at most one value, being the same for all $r' \geq r$. But by taking larger radii, every other value must be taken infinitely often, as $|z| \to \infty$.

□

The result above in fact generalizes to show that if $f : \mathbb{D} \setminus \{0\} \to X$ is a holomorphic map to a compact hyperbolic Riemann surface, then it extends to $f : \mathbb{D} \to X$. Moreover, if $X' = X \setminus \{p_1, \ldots, p_n\}$ is a hyperbolic Riemann surface of finite type, that is, X' is compact with a finite number of punctures, then any holomorphic map $f : \mathbb{D} \setminus \{0\} \to X'$ extends to a holomorphic map $f : \mathbb{D} \to X$, where it either extends to X' itself or to one of the punctures.

3 Normal Families: Montel's Theorem

Recall the definition of a normal family according to Milnor: for S and T Riemann surfaces, a family $\mathcal{F} \subseteq C(S,T)$ is *normal* if every sequence $(f_n) \subset \mathcal{F}$ either has a subsequence which converges locally uniformly to some function (which must be continuous), or has a subsequence which *diverges locally uniformly*, that is, for the subsequence f_{n_j} and all compact subsets $K \subseteq S$, $K' \subseteq T$, there exists $N \in \mathbb{N}$ such that for all $j \geq N$, $f_{n_j}(K) \cap K' = \emptyset$. In other words, the images of every compact subset of S eventually escape any other given compact subset of T.

As the space $C(S,T)$, which contains $\mathrm{Hol}(S,T)$ as a closed subspace, is Hausdorff and metrizable with respect to the topology of locally uniform convergence (because S is σ-compact), sequential compactness and compactness are equivalent notions in $C(S,T)$. Hence a subset $\mathcal{G} \subset \mathrm{Hol}(S,T)$ is precompact if and only if every sequence in $\mathcal{G}$ has a locally uniformly convergent subsequence. Note that this excludes the case of locally uniform divergence as above, which allows for more families to be normal. (We note that Ahlfors' more general definition also allows normal families to have subsequences diverging locally uniformly to infinity.)

Problem (3-e. Local Normality)**.** Show that normality is a local property. More precisely, let S and T be any Riemann surfaces, and let $\{f_\alpha\}$ be a family of holomorphic maps from S to T. If every point of S has a neighborhood U such that the collection $\{f_\alpha|_U\}$ of restricted maps is a normal family in $\mathrm{Hol}(U,T)$, show by a diagonal argument, similar to the proof of 3.2, that the family $\{f_\alpha\}$ is itself normal.

Proof. Let $\{f_n\}$ be a sequence in the family $\{f_\alpha\}$, which we want to prove either has a subsequence which converges locally uniformly, or has a subsequence which diverges locally uniformly to infinity in T.

Fix a countable dense subset $\{s_j\}$ of S. For all s_j, we have a neighborhood U_j such that the family $\{f_\alpha|_{U_j}\}$ is normal. In particular, there is a subsequence (f_{n_k}) of the (f_n) such that $(f_{n_k}|_{U_1})$ either converges locally uniformly, or diverges locally uniformly. If it converges locally uniformly, call $g_1 : U \to T$ the holomorphic limit. Considering now U_2, there is some subsequence $(f_{n_{k_l}})$ of (f_{n_k}) that either converges locally uniformly in U_2 to some g_2, or diverges locally uniformly. Proceeding by induction and taking a diagonal subsequence, we obtain a subsequence (h_n) of (f_n) such that, for each j, $(h_n|_{U_j})$ either converges locally unformly to some holomorphic function $g_j : U_j \to T$, or diverges locally uniformly.

Let $V \subseteq S$ be the set of points p for which $h_n(p)$ diverges to infinity. By the above construction, V is open, where if $p \in V$ then p belongs to some U_j where $h_n|_{U_j}$ diverges locally uniformly. Moreover, it is also closed; Suppose $p_i \to p$, where $h_n(p_i) \to \infty$. Then $p \in U_j$ for some U_j, and eventually the points p_i are contained in U_j. if $h_n|_{U_j} \to g_j$ locally uniformly, then $h_n(p_i) \to g_j(p_i)$, which cannot be the case as $h_n(p_i) \to \infty$. Hence $h_n|_{U_j}$ diverges locally uniformly, and $h_n(p) \to \infty$.

The above implies that V is open and closed, and since we assume S is a connected Riemann surface, either $V = \emptyset$ or $V = S$. Suppose first that $V = \emptyset$, so that for all j, $h_n|_{U_j} \to g_j$ locally uniformly. Because for every $p \in S$ the limit $\lim_{n\to\infty} h_n(p)$ is unique if it exists, We have a globally defined function $g : S \to T$ that is the pointwise limit of h_n, and $g|_{U_j} = g_j$. Hence g is holomorphic. Moreover, for $p \in S$, we know that $p \in U_j$, and for some compact neighborhood $N_j \subset U_j$ of p, $h_n|_{N_j} \to g_j|_{N_j} = g|_{N_j}$ uniformly, so that h_n converges locally uniformly to g.

Now assume $V = S$. Consider $K \subseteq S$ and $K' \subseteq T$ compact subsets. Letting N_j be compact neighborhoods of the s_j in U_j, there are finitely many N_j covering K. By definition of locally uniform divergence, there exists n_j such that, for all $n \geq n$, $h_n(N_j) \cap K' = \emptyset$. Taking the maximum $n_{\max}$ over all the finitely many n_j, we have consequently that $h_n(K) \cap K' = \emptyset$ for $n \geq n_{\max}$, hence the h_n diverge locally uniformly. □

An observation to the above proof is that the hypothesis that S be connected is necessary, otherwise we could consider a family of holomorphic functions $f_n \sqcup g_n : S_1 \sqcup S_2 \to T$ which is locally uniformly convergent on S_1, but locally uniformly divergent on S_2, and the family will be neither on $S_1 \sqcup S_2$. Hence, in more generality, if the family is normal on a neighborhood of every point, it will be normal on every connected component of S.

The above complications naturally vanish in the case of T being compact, where we obtain the corrolary:

Corollary 3.1. *If T is compact and $U \subseteq S$ is open, then $\mathcal{F}|_U \subset \mathrm{Hol}(U,T)$ is normal if and only if, for all $z \in U$, there is some neighborhood U_z of z in U such that $\mathcal{F}|_{U_z} \subset \mathrm{Hol}(U_z,T)$ is normal.*

This corollary can be used later to "piece together" normality of open subsets to invoke normality on bigger ones, even if disconnected.

Problem (3-f. Normality and Derivataives)**.** Let $f : S \to T$ be holomorphic. Given Riemannian metrics on the Riemann surfaces S and T, we can define the *norm* of the derivative at a point $s \in S$ to be the real number $\|f'(s)\| \geq 0$ such that the induced linear mapping from T_sS to $T_{f(s)}T$ carries vectors of length 1 to vectors of length $\|f'(s)\|$. If T is compact, show that a family $\mathcal{F}$ of maps $f : S \to T$ is normal if and only if the collection of norms $\|f'(s)\|$ is uniformly bounded as f varies over $\mathcal{F}$ and s varies over any compact subset of S.

Proof. Suppose $\mathcal{F} \subseteq \mathrm{Hol}(S,T)$ is normal. We want to show that, for any compact set K,

$$\sup_{s\in K,\ f\in\mathcal{F}} \|f'(s)\| < +\infty.$$

Suppose that there exists a compact set K such that this supremum is not finite, so there exists a sequence $(f_n) \subset \mathcal{F}$ and $(s_n) \subset K$ such that $\|f_n'(s_n)\| \to \infty$. By taking subsequences, we may assume that $f_n \to f$ locally uniformly, and that $s_n \to s \in K$. Let U be a compact coordinate neighborhood of s, where we may also assume $s_n \in U$ for all n. By continuity of the norm, there exists $M < +\infty$ such that, for all $p \in U$, $\|f'(p)\| < M$. As $f_n|_U \to f$ uniformly, and since uniform convergence of holomorphic functions implies uniform convergence of derivatives, we have that $\|f_n'\| \to \|f'\|$ uniformly in U (?). This contradicts that $\|f_n'(s_n)\| \to \infty$.

Conversely, suppose that over any compact set, $\|f'(s)\|$ is uniformly bounded for all $f \in \mathcal{F}$. Let $N \subseteq S$ be a compact connected (normal geodesic) neighborhood of some $p_0 \in S$, and $(f_n) \subset \mathcal{F}$ a sequence. Since $\sup_{f\in\mathcal{F},\ s\in N} \|f'(s)\|$ is bounded above by some λ, We have that if γ is a constant speed path from p_0 to p in γ, we have that

$$L(f \circ \gamma) = \int_\gamma \|f'(\gamma(t))\| dt \leq \lambda L(\gamma).$$

Hence $d(f(p), f(p_0))$ is bounded above by some constant D, for all $p \in N$ and $f \in \mathcal{F}$. As T is compact, $(f_n(p))$ has a convergent subsequence $f_{n_k}(p) \to$

$q \in T$. The above reasonings imply that all f_{n_k}, for sufficiently large k, map the compact set N into the compact set $\overline{B_{D+\varepsilon}(q)}$ in T. Hence the family $\{f_{n_k}\}$ is normal, and has a locally uniformly convergent subsequence. □

Theorem 3.2. *Let S and T be Riemann surfaces, where S is connected. Let $f_n : S \to T$ be a sequence of continuous functions that diverge locally uniformly to infinity in T. Then there exists a subsequence (f_{n_k}) and an end $e \in \mathrm{Ends}(T)$ such that (f_{n_k}) converges uniformly to e on compact subsets.*

Proof. Let (K_i) be a compact exhaustion of S by connected full compact sets, and analogously (L_j) of T. By definition, given i and j, there exists some $N = N_{i,j} \in \mathbb{N}$ such that, for all $n \geq N$, $f(K_i) \in T \setminus L_j$. As K_i is connected, $f(K_i)$ also is, so K_i gets mapped by f_n to a single component $U^n_{i,j}$ of $T \setminus L_j$. Since there are finitely many components of $T \setminus L_j$, by taking a subsequence of the f_n, we may assume that this component $U_{i,j} = U^n_{i,j}$ is the same for all $n \geq N_{i,j}$.

Consider $i' > i$. For this subsequence of the f_n, by connectedness of $K_{i'}$, for all big n the component of $T \setminus L_j$ which $f_n(K_{i'})$ belongs to must be the same $U_{i,j}$. Hence starting from $i = 1$, we may take $U_j = U_{1,j}$ to be a component of $T \setminus L_j$ defined by the compact exhaustion for which eventually all points of S get mapped to.

Now, if $j' > j$, by taking another subsequence of the f_n, we produce a component $U_{j'}$ of $T \setminus L_{j'}$ such that $U_{j'} \subseteq U_j$, and all of S eventually gets mapped to $U_{j'}$ (uniformly on compact sets). By proceeding inductively, taking subsequences and finally taking a diagonal subsequence, we obtain a decreasing chain of components of the $T \setminus L_j$, which in fact defines an end. □

4 Fatou and Julia: Dynamics on the Riemann Sphere

Lemma 4.1 (Details on the Invariance Lemma)**.** *If $f : S \to S$ is a holomorphic map on a compact Riemann surface S , the Julia set $J(f)$ is fully invariant, that is, $z \in J \iff f(z) \in J$, or equivalently, $f^{-1}(J) = J$.*

Proof. It is equivalent to show that the fatou set F is fully invariant. Suppose $U \subseteq F$ is a connected open set; we show that $\{f^n|_{f^{-1}(U)}\}$ is a normal family, which will imply that $f^{-1}(U) \subseteq F$. Let (f^{n_j}) be any sequence of iterates, and consider the sequence $(f^{n_j - 1})$, which has a subsequence $(f^{n_{j_k}-1})$ that converges locally uniformly in U to some g. Let $K \subseteq f^{-1}(U)$ be compact, so that $f(K) \subseteq U$ is compact. Then, fixing a metric on S,

$$\begin{aligned}\sup_{x\in K} d(f^{n_{j_k}}(x), (g\circ f)(x)) &= \sup_{x\in K} d(f^{n_{j_k}-1}(f(x)), g(f(x))) \\ &= \sup_{y\in f(K)} d(f^{n_{j_k}-1}(y), g(y)) \to 0,\end{aligned}$$

so that the subsequence $f^{n_{j_k}}$ converges uniformly in K to $g\circ f$. Hence $(f^{n_{j_k}})$ converges locally uniformly to $g \circ f$ in $f^{-1}(U)$.

Now let $z \in F$ and U a connected precompact neighborhood of z such that $\{f^n|_U\}$ is a normal family. As f is open, $f(U)$ is a connected precompact neighborhood of $f(z)$. Moreover, we may assume $f|_U : U \to f(U)$ is a branched finite covering map. Let (f^{n_j}) be some sequence of iterates, and consider (f^{n_j+1}). Then there is some subsequence $(f^{n_{j_k}+1})$ that is locally uniformly convergent on U to some g. Pointwise, for all $x \in U$,

$$f^{n_{j_k}+1}(x) = f^{n_{j_k}}(f(x)) \to g(x).$$

Now, if $y \in f(U)$, then $y = f(x)$ for $x \in U$, and

$$f^{n_{j_k}}(y) = f^{n_{j_k}}(f(x)) \to g(x).$$

Hence the functions $f^{n_{j_k}}$, in $f(U)$, converge pointwise to a function h such that $h\circ f = g$. By our hypotheses on U, if $K \subseteq f(U)$ is compact, $(f|_U)^{-1}(K)$ is compact. Therefore if $y \in K$, by taking $f(x) = y$,

$$d(f^{n_{j_k}}(y), h(y)) = d(f^{n_{j_k}+1}(x), g(x)) \to 0$$

uniformly for $x \in (f|_U)^{-1}(K)$, so that $f^{n_{j_k}}$ converges to h uniformly in K. As this compact K is arbitrary, we have locally uniform convergence, and then $f(z) \in F$, completing the proof of complete invariance. □

Lemma 4.2 (Details on the Iteration Lemma)**.** *For any $k > 0$, the Julia set $J(f^k)$ coincides with $J(f)$.*

Proof. Naturally $F(f) \subseteq F(f^k)$, since if U is an open set such that $\{f^n|_U\}$ is normal, and (f^{kn_j}) is any sequence in the family $\{f^{kn}\}$, then it is a sequence in the family $\{f^n\}$, and therefore has a locally uniformly convergent subsequence. Hence $\{f^{kn}|_U\}$ is normal.

Now suppose $z \in F(f^k)$, and U is a connected neighborhood of z on which $\{f^{kn}|_U\}$ is a normal family. Then $\{f^{kn}|_U\}$ is contained in a compact set $K \subseteq \mathrm{Hol}(U, S)$. Post-composition with f in $\mathrm{Hol}(U, S)$ is a continuous operation, so $f^i \circ K$ is compact in $\mathrm{Hol}(U, S)$. Then every iterate of f is contained in the set

$$K \cup (f \circ K) \cup (f^2 \circ K) \cup \ldots \cup (f^{k-1} \circ K) \subset \mathrm{Hol}(U, S),$$

and the family $\{f^n|_U\}$ is normal. Therefore $z \in F(f)$. □

Note the following fact:

Lemma 4.3. *If $f : S \to S$ is holomorphic and $g \in \mathcal{G}(S)$, then we have a homeomorphism*

$$J(f) \longleftrightarrow J(g \circ f \circ g^{-1})$$

given by $z \mapsto g(z)$.

Proof. Note that $(gfg^{-1})^n = gf^ng^{-1}$, and the family $\{gf^ng^{-1}|_U\}$ is normal for an open set U if and only if $\{f^n|_{g(U)}\}$ is normal, because a sequence of iterates f^{n_j} converges uniformly on some compact set $g(K)$ to h if and only if $gf^{n_j}g^{-1}$ converges uniformly to ghg^{-1} on K. □

Problem (4-a. Degree One)**.** If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is rational of degree $d = 1$, show that the Julia set $J(f)$ is either vacuous, or consists of a single repelling or parabolic fixed point.

Proof. As f is rational of degree $d = 1$, we have that $f \in \mathcal{G}(\hat{\mathbb{C}})$. Every Möbius transformation has either a single fixed point or two fixed points. By conjugating f by some other Möbius transformation taking the fixed points to 0 and ∞ or to only ∞, we may assume that f is of the form $z \mapsto \lambda z$ or $z \mapsto z + b$, where $\lambda, b \in \mathbb{C}$. In the first case, by possibly conjugating the map by $1/z$, we may assume $|\lambda| \leq 1$. If $|\lambda| < 1$, 0 is an attracting fixed point with multiplier λ, and any compact set in $\mathbb{C}$ converges to 0 uniformly, where ∞ is the unique point in the Julia set, being a repelling fixed point.

If on the other hand $|\lambda| = 1$, f is a rotation about 0. If $\lambda = e^{2\pi i p/q}$, then some iterate of f is the identity and the Julia set is empty. If $\lambda = e^{2\pi i\theta}$ where $\theta \notin \mathbb{Q}$, then

$$f^n(z) = \lambda^n z = e^{2\pi i n\theta} z,$$

and we show that $F(f) = \hat{\mathbb{C}}$. This is because λ^n is dense in S^1, hence, if (f^{n_j}) is a subsequence of iterates, there is some subsequence $f^{n_{j_k}}$ such that $\lambda^{n_{j_k}} \to \mu \in S^1$. Therefore for $K \subset \mathbb{C}$ compact,

$$d(\lambda^{n_{j_k}} z, \mu z) \leq |z||\lambda^{n_{j_k}} - \mu| \leq C|\lambda^{n_{j_k}} - \mu| \to 0$$

uniformly. That $\infty \in F(f)$ is easy to see from an uniformizing parameter around ∞.

If $f(z) = z + b$, by conjugating z by z/b we may assume $f(z) = z + 1$. Any compact set in $\mathbb{C}$ converges uniformly under iteration to ∞, but any neighborhood of infinity has points which first repel from ∞ before converging to it. We have seen that a parabolic fixed point belongs to the Julia set. □

Problem (4-b. Maps with grand orbit finite points)**.** Now suppose that f is a rational map of degree $d \geq 2$. Show that f is actually a polynomial if and only if $f^{-1}(\infty) = \{\infty\}$, so that the point at infinity is a grand orbit finite fixed point for f. Show that f has both zero and infinity as grand orbit finite fixed points if and only if $f(z) = \alpha z^n$, where $n = \pm d$ and $\alpha \neq 0$. Conclude that f has grand orbit finite points if and only if it is conjugate, under some factional linear change of coordinates, either to a polynomail or to the map $1/z^d$.

Proof. If f is a polynomial, it is clear that $\mathbb{C}$ is fully invariant by f, hence $\{\infty\}$ is too. Now suppose $f^{-1}(\infty) = \{\infty\}$. Then $\{\infty\}$ is a grand orbit finite fixed point for f, so that by previous results it must be a superattracting fixed point. In fact, since it has d pre-images counted with multiplicity, ∞ must be a critical point with multiplicity $d - 1$.

If
$$f(z) = \frac{p(z)}{q(z)} = \frac{a_m z^m + \ldots + a_0}{b_n z^n + \ldots + b_0}$$
is a ratio of polynomials with no common factors and $a_m, b_n \neq 0$, Then each root of $q(z)$ must be mapped by f to infinity. Since no point in $\mathbb{C}$ can be mapped to ∞, it follows that q is a constant polynomial, and f is a a polynomial.

If 0 and ∞ are both grand orbit finite points for f, they must be the only ones from previous results. Hence $f^{-1}(\{0, \infty\}) \subseteq \{0, \infty\}$. We divide into cases. If $f(0) = f(\infty) = 0$, this means no point in $\mathbb{C}$ maps to infinity. As $f(\infty) = 0$, f is a bounded holomorphic map on $\mathbb{C}$, hence constant, a contradiction. An analogous contradiction arises when $f(0) = f(\infty) = \infty$ by conjugating the map with $1/z$.

Hence either 0 and ∞ are fixed points or form a periodic orbit of period 2. If they are fixed points, we have seen that f is a polynomial, and as 0 must be a critical point of multiplicity $d-1$, we have $f(z) = \alpha z^d$. On the other case, by composing f with $1/z$, we obtain that $f(1/z) = \alpha z^d$, so $f(z) = \alpha z^{-d}$.

If f is a rational map with grand orbit finite points, then there must be either one or two of them. Since the action of $\mathcal{G}(\hat{\mathbb{C}})$ is triply transitive, we may conjugate f b some map taking these points either to ∞ or to $\{0, \infty\}$, and apply the previous results. Conjugation from αz^{-d} to z^{-d} comes from conjugating the map by λz, where λ is a $(d+1)$-th root of α^{-1}. □

Problem (4-c. Fixed point at infinity)**.** If f is a rational function with a fixed point at infinity, show that the multiplier λ at infinity is equal to $\lim_{z \to \infty} 1/f'(z)$. In particular, this fixed point is superattracting if and only if $f'(z) \to \infty$ as $z \to \infty$.

Proof. Consider the uniformization $z \mapsto 1/z$ around infinity, where we locally have the map

$$\frac{1}{f(\frac{1}{z})}$$

around 0. If f is a rational map

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_m z^m + \ldots + a_0}{b_n z^n + \ldots + b_0},$$

then

$$\frac{1}{f(\frac{1}{z})} = \frac{b_n(1/z^n) + \ldots + b_0}{a_m(1/z^m) + \ldots + a_0} = z^{m-n}\frac{b_n + \ldots + b_0 z^n}{a_m + \ldots + a_0 z^m},$$

so that if ∞ is a fixed point, since $a_m, b_n \neq 0$, we have $m - n \geq 1$. Taking the derivative at $z = 0$, we obtain that the multiplier at infinity is b_n/a_m, if $m - n = 1$, and 0, if $m - n \geq 2$. Now, the derivative of $f(z)$ is

$$\begin{aligned}
f'(z) &= \frac{p'(z)q(z) - p(z)q'(z)}{q(z)^2} \\
&= \frac{(ma_m z^{m-1} + \ldots + a_1)(b_n z^n + \ldots b_0) - (a_m z^m + \ldots + a_0)(nb_n z^{n-1} + \ldots + b_1)}{(b_n z^n + \ldots + b_0)^2} \\
&= \frac{(m-n)a_m b_n z^{m+n-1} + \ldots + a_1 b_0 - a_0 b_1}{b_n^2 z^{2n} + \ldots + b_0^2} \\
&= \frac{1/z^{m+n-1}}{1/z^{2n}} \frac{(m-n)a_m b_n + \ldots + (a_1 b_0 - a_0 b_1)\frac{1}{z^{m+n-1}}}{b_n^2 + \ldots + b_0^2 \frac{1}{z^{2n}}} \\
&= z^{1-(m-n)} \frac{(m-n)a_m b_n + \ldots + (a_1 b_0 - a_0 b_1)\frac{1}{z^{m+n-1}}}{b_n^2 + \ldots + b_0^2 \frac{1}{z^{2n}}},
\end{aligned}$$

so that the limit of $1/f'(z)$ as $z \to \infty$ is either b_n/a_m, if $m - n = 1$, or 0, if $m - n \geq 2$, coinciding with the previous calculations. □

Problem (4-d. Self-similarity)**.** Show that the set of z for which (J, z) is locally conformally isomorphic to (J, z_0) is everywhere dense in J unless the following very exceptional condition is satisfied: For every backwards orbit $\cdots \mapsto z_2 \mapsto z_1 \mapsto z_0$ under f which terminates at z_0, some z_j with $j > 0$ must be a critical point of f. As an example, for the map $f(z) = z^2 - 2$, show that this condition is satisfied for the endpoints $z_0 = \pm 2$. Similarly, show that it is satisfied for the point $z_0 = 0.8i$ of the map

$$g(z) = z^3 + \frac{12}{25}z + \frac{116}{125}i.$$

For any f, show that there can be only finitely many such exceptional points z_0.

Proof. Suppose that there exists some open set U such that for all $z \in U \cap J$, (z, J) is not locally conformally isomorphic to (z_0, J). As iterated preimages of z_0 are dense in J, there will exist $z' \in U \cap J$ such that $f^n(z') = z$, for some n. Hence the failure of $f^n : N \to N_0$ being a conformal isomorphism for some neighborhoods N and N_0 happens only if f^n is a critical point for f, so that some $f^k(z')$ for $0 \leq k < n$ is a critical point of f. Hence the this backwards orbit of z_0 has a critical point.

Suppose now $w \in J$ is such that $f^m(w) = z_0$. Iterated preimages of w are dense, so some preimage of w is contained in U. For local conformality to fail, we must similarly have that either $f^k(w)$ is a critical point for some $0 \leq k < m$, or this preimage of w is a critical point.

For the purposes of notation, let

$$S_{z_0} = \{z \in \mathbb{C} \colon \exists n \in \mathbb{N} \text{ such that } f^n(z) = z_0\}$$

be the set of all points in all backwards orbits of z_0, Ω_f be the set of critical points of f, $\Omega = \Omega \cap S_{z_0}$ and $P = \bigcup_{n \geq 1} f^n(\Omega)$ the postcritical set of Ω.

Suppose that there exists infinitely many points $w \in S_{z_0}$ such that w is not a critical point nor maps to a critical point before mapping to z_0. This will imply that some preimage of w is a critical point. We take a sequence (w_j) of pairwise distinct such points. As Ω is finite, by taking a subsequence we may assume that there exists a single $p' \in \Omega$ such that $f^{n_j}(p) = w_j$, where $n_j \geq 1$ is minimal. We must also have that the n_j are distinct, so $n_j \to \infty$. This implies that the postcritical set $P_{p'}$ is contained in S_{z_0}; for if $f^k(p') = u$, we have some $n_j > k$, so

$$f^{n_j - k}(u) = f^{n_j - k}(f^k(p')) = f^{n_j}(p') = w_j \in S_{z_0},$$

so $u \in S_{z_0}$. Moreover, the postcritical set of z_0 is also in S_{z_0}, hence z_0 must be a periodic point for f. Then for all $p \in \Omega$, p is preperiodic, so P is finite. However, the set of $w \in S_{z_0}$ such that w is not a critical point nor a preimage of a critical point before mapping to z_0 is contained in P, so we obtain a contradiction.

Therefore this set is finite. If there were an infinite backwards orbit $\cdots \mapsto z_2 \mapsto z_1 \mapsto z_0$ that does not contain a critical point, then there can be only finitely many distinct point in this backeards orbit. Immediately this implies that some point in it is periodic, hence z_0 is periodic, and all $q \in \Omega$ are preperiodic. Furthermore, let w be a point in this backwards orbit that repeats infinitely often, where it is then periodic. Let m be the (minimal) period if p, and suppose some $z_j = w$ for some $j > m$. Then, for $j' < j$, we have that $z_{j'} = w$ if and only if $j' \equiv j \mod m$, by minimality of the period. Hence, for $j' > j$, if $j' \not\equiv j \mod m$, then we must have that $z_{j'} \neq w$. In other words, w can appear in this sequence only "aligning" its period with j.

But as w appears infinitely often in the backwards orbit, it will appear for arbitrarily big $j' \equiv j \mod m$. And when it appears, for all $j'' < j'$, such that $j'' \equiv j \mod m$, we will have that $z_{j''} = w$. All in all, this implies that for all $k \in \mathbb{N}$, $z_{j+km} = w$, and therefore this backwards orbit is periodic. Moreover, as w must map to z_0 under some iterate, z_0 will appear in this backwards orbit, and it will in fact correspond to the periodic orbit of z_0.

In summary, there can be at most one infinite backwards orbit of z_0 that contains no critical points. If it exists, z_0 is periodic, this backwards orbit will correspond to the periodic orbit of z_0, and all $p \in \Omega$ are preperiodic. All other preimages of z_0 will eventually have a critical point in its backwards orbit. In this case, the density of points locally conformally equivalent to (z_0, J) evidently fails.

Let $p \in \Omega$ be such that it is the first critical point to appear in some backwards orbit of z_0, that is, if n is the minimal n such that $f^n(p)$, then $f^i(p)$ is not critical for $0 < i < n$.

If z_0 is not periodic, then from the previous finiteness results, and the fact that any point has at most $\deg f$ preimages, there will exist some iterate $g = f^l$ of f such that all preimages of z_0 are critical points for g. In order for density of points locally conformally equivalent to (z_0, J) to fail, we must have that either z_0 maps to a critical point, or that its forward orbit is not dense in J.

If $A = \overline{\mathcal{O}^+(z_0)}$, then A is a forward invariant closed subset of J. Must A be finite?

Consider $f(z) = z^2 - 2$, where 0 is the unique critical point, $f(0) = -2$, $f(-2) = f(2) = 2$. Therefore $f^{-1}(2) = \{-2, 2\}$ and $f^{-1}(-2) = \{0\}$, so that for any (infinite) backwards orbit of $z_0 = -2$, z_1 is 0, a critical point, but for $z_0 = 2$, we have the infinite backwards orbit $z_j = 2$ for all j, where no point is critical. As 2 is a fixed point for f, we cannot have any other (z, J) be locally conformally isomorphic to $(2, J)$, for either we have an iterate of 2 mapping to z, so $z = 2$, or an iterate of z mapping to 2, so either $z = 2$ or z passes through a critical point to map to 2.

For $g(z)$, note that $g'(z) = 3z^2 + 12/25$, so the critical points are $\pm\frac{2}{5}i$. If $z_0 = 4i/5$, we have that

$$g\left(\frac{2}{5}i\right) = \frac{132}{125}i, \quad g\left(-\frac{2}{5}i\right) = \frac{4}{5}i.$$

Since we want to solve $g(z) = 4i/5$, we obtain

$$g(z) - \frac{4}{5}i = \left(z + \frac{2}{5}i\right)\left(z^2 - \frac{2}{5}iz + \frac{8}{25}\right),$$

so that the two other preimages of z_0 are $\frac{4}{5}i$ and $-\frac{2}{5}i$. In fact, the critical point $-\frac{2}{5}i$ maps to z_0 with multiplicity 2, and z_0 is also a fixed point. The same analysis as above applies to z_0 for some (z, J) to be locally conformally isomorphic to (z_0, J).

We show there can be at most finitely many points for which density of (z, J) locally conformally equivalent to (z_0, J) fails. We have to understand the case where z_0 is possibly not preperiodic, but every preimage of z_0 is critical; why is that a problem? (incomplete: may revisit question later. It is more complicated than anticipated.)

□

Problem (4-e. A Cantor Julia set.)**.** If $f(z) = z^2 - 6$, show that $J(f)$ is a Cantor set contained in the intervals $[-3, -\sqrt{3}] \cup [\sqrt{3}, 3]$. More precisely, show that a point in $J(f)$ with orbit $z_0 \mapsto z_1 \mapsto \cdots$ is uniquely determined by the sequence of signs $\epsilon_j = z_j/|z_j| = \pm 1$. In fact

$$z_0 = \epsilon_0 \sqrt{6 + \epsilon_1 \sqrt{6 + \epsilon_2 \sqrt{6 + \cdots}}}.$$

Show that every orbit outside of this Cantor set must escape to infinity.

Proof. Note that $f(3) = f(-3) = 3$, and $f(-\sqrt{3}) = f(\sqrt{3}) = -3$. Restricting ourselves to the real axis $\mathbb{R}$ for the moment, we see that f maps the interval $[-3, -\sqrt{3}]$ to $[-3, 3]$ monotonically and likewise maps $[\sqrt{3}, 3]$ to $[-3, 3]$ monotonically. On these these two intervals, the derivative of f is $|f'(z)| > 1$. Classical arguments in real unimodal dynamicals system show us that the set of all points $x \in \mathbb{R}$ that do not escape $[-3, 3]$ under iteration forms a Cantor set, contained in $[-3, -\sqrt{3}] \cup [\sqrt{3}, 3]$, where each point is determined uniquely by an itinerary through these two intervals, where an explicit expression is as above. The dynamical system on this Cantor set is topologically conjugate to the shift map on $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$, and points outside of it diverge to infinity.

Periodic points are dense in this cantor set Λ, and since the derivative of f is strictly greater than 1 on Λ, all of them are repelling. Therefore they belong to the Julia set, and the closure belongs to $J(f)$, so $\Lambda \subseteq J(f)$. Another argument, more in line in the book, is that the two branches of f^{-1}, $\pm\sqrt{z+6}$, are strictly contracting on $[-3, 3]$ and carry the interval to $[-3, -\sqrt{3}]$ or to $[\sqrt{3}, 3]$.

We see that Λ is a compact fully invariant subset of $J(f)$. For $z \in \Lambda$, we see that its set of preimages is contained in Λ and dense in $J(f)$, hence Λ is dense in $J(f)$. (This argument shows that if $S \subseteq J(f)$ and $f^{-1}(S) \subseteq S$, then $\overline{S} = J(f)$.

Moreover, we see that ∞ is a superattracting fixed point for f, as it is a polynomial. Hence the basin of attraction of ∞ is contained in $F(f)$, and contains the exterior of some disk $\mathbb{D}_R$. Suppose U is a Fatou component disjoint from $\mathcal{A}(\infty)$, so in particular $U \subset \mathbb{D}_R$. We know that $\partial U \subseteq J(f) = \Delta$. We know that Imz attains a maximum and minimum on the boundary of U, but since all points the Julia set have imaginary part 0, we see that $Imz = 0$ for all $z \in U$. This is a contradiction with U open, hence $U = \emptyset$, and $F(f) = \mathcal{A}(\infty)$. □

(General question: can a bounded open set U in $\mathbb{R}^2$ have its boundary be contained in a cantor set Λ?)

Problem (4-h. Liapunov Stability)**.** A point $z_0 \in \hat{\mathbb{C}}$ is *stable in the sense of Liapunov* for a rational map f if the orbit of any point which is sufficiently close to z_0 remains uniformly close to the orbit of z_0 for all time. More precisely, for every $\varepsilon > 0$, there should exist a $\delta > 0$ so that if z has spherical distance $\sigma(z, z_0) < \delta$ then $\sigma(f^n(z), f^n(z_0)) < \varepsilon$ for all n. Show that a point is Liapunov stable if and only if it belongs to the Fatou set.

Proof. We recall Arzela-Ascoli's theorem:

Theorem 4.4. *For a family of continuous functions $\mathcal{F} : \Omega \subseteq \mathbb{C} \to Y$, where Y is a complete metric space, $\mathcal{F}$ is normal if and only if:*

- *$\mathcal{F}$ is uniformly equicontinuous on compact subsets $K \subseteq \Omega$;*
- *for each $z \in \Omega$, $\mathcal{F}(z) = \{f(z) \colon f \in \mathcal{F}\}$ is precompact in Y.*

A small comment is that normality of holomorphic maps between hyperbolic Riemann surfaces comes from Arzela-Ascoli and Pick's theorem, where the second condition above decides if the family is locally uniformly divergent or not.

Suppose that z_0 is in the Fatou set, so that for some connected neighborhood U if z_0, $\{f^n|_U\}$ is a normal family. For $Y = \hat{\mathbb{C}}$ with the spherical metric (being compact so we do not have the issue of locally uniform divergence to infinity), equicontinuity of $\{f^n|_U\}$ on compact subsets of U implies that, for $\overline{B_{\delta'}(z_0)} \subset U$ compact, for all $\varepsilon > 0$ there is a $\delta'' > 0$ such that for all $n \in \mathbb{N}$ and $z, z' \in \overline{B_{\delta'}(z_0)}$, if $\sigma(z, z') < \delta''$ (in the spherical metric restricted to U, given that they are equivalent), then

$$\sigma(f^n(z), f^n(z')) < \varepsilon.$$

In particular, $\sigma(f^n(z), f^n(z_0)) < \varepsilon$, so we need only take $\delta = \min\{\delta', \delta''\}$. Hence z_0 is Liapunov stable.

Conversely, if z_0 is Liapunov stable, then

$$S_\varepsilon = \bigcap_{n=0}^{\infty} f^{-n}(B_\varepsilon(f^n(z_0)))$$

contains z_0 in its interior, for all $\varepsilon > 0$. To prove normality of $\{f^n|_U\}$ on some neighborhood of z_0, we need only find a neighborhood of z_0 on which $\{f^n|_U\}$ is uniformly equicontinuous. For some compact neighborhood $\overline{B_{\delta'}} = \overline{B_{\delta'}(z_0)} \subset U$, and for $\varepsilon > 0$, Let $N = S_{\varepsilon/2} \cap \overline{B_{\delta'}}$. Then, for all n and $z, z' \in N$, we have that

$$\sigma(f^n(z), f^n(z')) \leq \sigma(f^n(z), f^n(z_0) + \sigma(f^n(z_0), f^n(z')) < \varepsilon.$$

Hence $\{f^n\}$ is uniformly equicontinuous on N. □

Problem (4-i. Fatou components)**.** If Ω is a connected component of the Fatou set of f, show that $f(\Omega)$ is also a connected component of $F(f)$.

Proof. Naturally $f(\Omega)$ is open, connected and contained in the fatou set $F(f)$, hence contained in a single Fatou component Ω'. Note that $\partial\Omega \subseteq J(f)$, so $f(\partial\Omega) \subset J(f)$ and $f(\partial\Omega)$ is disjoint from Ω'. More precisely, as f is continuous, $f(\overline{\Omega}) \subseteq \overline{f(\Omega)}$, hence

$$f(\partial\Omega) \subseteq \overline{f(\Omega)} = f(\Omega) \cup \partial f(\Omega).$$

As $f(\Omega)$ is in the Fatou set, we must have that $f(\partial\Omega) \subseteq \partial f(\Omega)$, and since $\overline{f(\Omega)} \subseteq \overline{\Omega'}$, we also have $f(\partial\Omega) \subseteq \partial\Omega'$.

We show $f|_\Omega : \Omega \to \Omega'$ is a proper map. If $K \subseteq \Omega'$ is compact, then let $(x_n)_n$ be a sequence in $(f|_\Omega)^{-1}(K)$. Then a subsequence (x_{n_k}) is such that $f(x_{n_k}) \to y \in K$. If (x_{n_k}) has no convergent subsequence, then it must escape every compact set of Ω. As $\overline{\Omega}$ is compact (here we are assuming that $f : S \to S$, where S is compact), (x_{n_k}) has a subsequence $x_{n_{k_l}} \to x \in \partial\Omega$. But then $f(x) = y \in f(\partial\Omega) \subseteq \partial\Omega'$, a contradiction. Hence $f^{-1}(K)$ is compact, and $f|_\Omega : \Omega \to \Omega'$ is a proper map.

As Ω' is locally compact and Hausdorff, $f|_\Omega$ is a closed map, so $f|_\Omega(\Omega) \subseteq \Omega'$ is open and closed. But then $f(\Omega) = \Omega'$, as we wanted to show. We also proved that a map between two Fatou components, when S is compact, is always proper. □

The hypothesis that S be compact above is necessary, as we may consider the map $z \mapsto \lambda z$ on $\mathbb{D}$, when $|\lambda| < 1$, or $z \mapsto z + i$ in $\mathbb{H}$.

5 Dynamics on Hyperbolic Surfaces

Problem (5-b. Accumulation points of a path)**.** In any Hausdorff space X, show that the closure of a connected set is connected, and show that the intersection of any nested sequence $K_1 \supset K_2 \supset \cdots$ of compact connected sets is again connected. Now consider an infinite path $p : [0, \infty) \to X$ in a compact Hausdorff space. Show that the set of all accumulation points of $p(t)$ ast $t \to \infty$ can be identified with the intersection of closures

$$\bigcap_t \overline{p[t, \infty)},$$

and therefore is a non-vacuous compact connected set.

Proof. We will prove the stronger fact: If $C \subseteq X$ is connected and $C \subseteq D \subseteq \overline{C}$, then D is connected. Suppose $D \subseteq U \cup V$, where U and V are disjoint and open in X. Then $C \subseteq U \cup V$, and as C is connected, without loss of generality $C \subseteq U$ and $C \cap V = \emptyset$, so $V \subseteq X \setminus C$.

As U and V are also closed in X, and

$$\overline{C} = \bigcap_{C \subseteq F} F$$

is the intersection of all closed subsets containing C, we have that $D \subseteq \overline{D} \subseteq U$, and therefore $D \cap V = \emptyset$. This concludes that D is connected.

Any closed subset of a compact set is compact, and in a Hausdorff space, compact sets are always closed. Hence the intersection $\bigcap K_i$ is compact and non-empty; if it were empty, $K_1 \setminus \bigcap_{i \geq n} K_i$ is an open cover of K_1, hence has a finite subcover, which would imply that some $K_i = \emptyset$, which we exclude as an assumption.

We show it is connected. If $K = \bigcap K_i \subseteq U \cup V$ where U and V are open and disjoint, we claim there is some K_i such that $K_i \subseteq U \cup V$. If that were not the case, denoting $W = U \cup V$ as an open set, we have that for all i, $K_i \setminus W$ is a nonempty compact set. But the

$$\bigcap_i (K_i \setminus W) = \left(\bigcap_i K_i \right) \setminus W = K \setminus W$$

is nonempty, a contradiction. Hence for some K_i, $K_i \subseteq U \cup V$. As K_i is connected, without loss of generality $K_i \subseteq U$ and $K_i \cap V = \emptyset$, so that $K \subseteq U$ and $K \cap V = \emptyset$, and K is connected.

An observation in the above proof is that we do not in fact need all of the K_i to be connected, but only infinitely many.

For a path $p : [0,\infty) \to X$ on X compact Hausdorff, we know that $[t,\infty)$ is connected for all t, and therefore also $p[t,\infty)$. We see that $\overline{p[t,\infty)}$ is connected and compact, as a closed subset of a compact space, Hence the intersection

$$K = \bigcap_t \overline{p[t,\infty)}$$

is compact, connected and nonempty. If $x \in K$, then by definition, for all n, there is some sequence $(t_j^n)_j$ such that $t_j^n \geq n$ and $p(t_j^n) \to x$. By taking a diagonal subsequence, we see that $p(t_n^n) \to x$, and $t_n^n \to \infty$, so that x is an accumulation point of the path as $t \to \infty$.

Conversely, if there is some sequence $(t_j)_j$ such that $t_j \to \infty$ and $p(t_j) \to x$, Then for all t, we may take a subsequence so that all the terms satisfy $t_j \geq t$, so $x \in \overline{p[t,\infty)}$. As this is true for all t, we have $x \in K$.

□

6 Dynamics on Euclidean Surfaces

Problem (6-a. The derivative of a torus map)**.** Consider the torus $\mathbb{T} = \mathbb{C}/\Lambda$, where we may assume that $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ with $\tau \notin \mathbb{R}$. Given $\alpha \in \mathbb{C}$, show that there exists a holomorphic map $f(z) \equiv \alpha z + c$ from $\mathbb{T}$ to itself if and only if $\alpha\Lambda \subset \Lambda$, or in other words if and only if both α and $\alpha\tau$ belong to Λ. Show that an arbitrary integer $\alpha \in \mathbb{Z}$ will satisfy this condition. On the other hand, show that there exists such a map with derivative $\alpha \notin \mathbb{Z}$ if and only if α satisfies a quadratic equation of the form

$$\alpha^2 + p\alpha + d = 0,$$

where $d = |\alpha|^2$ is the degree, and where p is an integer with $p^2 < 4d$. For a map of degree $d = |\alpha|^2 = 1$ show that α must be an m-th root of unity with $m = 1, 2, 3, 4$, or 6. If $m \neq 1$, conclude that f^m must be the identity map. Show that the cases $m = 3, 4, 6$ occur for suitably chosen lattices, and that the case $m = 1, 2$ occur for an arbitrary lattice. In the special case $\alpha = 1$, show that the closure of every orbit under f is either a finite set, a finite union of parallel circles, or the full torus $\mathbb{T}$.

Proof. Note that in order to be well defined, we need to have that $f(z+\omega) \sim f(z)$, for all $\omega \in \Lambda$; therefore $\alpha z + \alpha\omega + c \sim \alpha z + c$, and so $\alpha\omega \in \Lambda$, which implies $\alpha\Lambda \subseteq \Lambda$. Naturally for $\alpha \in \mathbb{Z}$, we have that $\alpha\omega \in \Lambda$, due to the abelian group structure of Λ.

Now, $\alpha\Lambda \subseteq \Lambda$ if and only if

$$\begin{cases} \alpha 1 = m + n\tau, \\ \alpha\tau = p + q\tau, \end{cases}$$

for $m, n, p, q \in \mathbb{Z}$. Moreover, $n = 0 \iff \alpha \in \mathbb{Z}$. So if $\alpha \notin \mathbb{Z}$, then $\tau = (\alpha - m)/n$, and therefore

$$\begin{aligned} \alpha\frac{\alpha - m}{n} &= p + q\frac{\alpha - m}{n} \\ \implies \alpha^2 - m\alpha &= np + q\alpha - qm \\ \implies \alpha^2 - (m+q)\alpha + qm - np &= 0 \\ \implies \alpha^2 - (\operatorname{tr} M)\alpha + \det M &= 0, \end{aligned}$$

where M is the matrix of the linear transformation with respect to the basis $\{1, \tau\}$ that multiplication by α induces on the lattice Λ. Note that $\det M$ and $\operatorname{tr} M$ do not depend on the basis we choose to represent multiplication by α, and $\det M$ is the degree of f, hence $\det M = |\alpha|^2$. Note then that

$$\alpha = \frac{\operatorname{tr} M \pm \sqrt{(\operatorname{tr} M)^2 - 4 \det M}}{2},$$

and $\alpha \in \mathbb{R}$, we would have that $n = 0$, so $\alpha \in \mathbb{Z}$, which is not what we assumed. Hence, for α to be non-real, we need that $(\operatorname{tr} M)^2 < 4 \det M$.

If $|\alpha|^2 = 1$, so that $\det M = 1$, we have that $M \in \mathrm{SL}_2(\mathbb{Z})$ when computed with respect to the basis $\{1, \tau\}$ of Λ, so that multiplication by α on Λ has an invers, where it must be multiplication by α^{-1}.

If $|\alpha|^2 = 1$, consider the case $\alpha \in \mathbb{Z}$, so that $\alpha = \pm 1$. Naturally then α is either 1 a 2-nd root of unity, and every lattice admits the maps $f(z) = z$ and $f(z) = -z$.

Assume now $|\alpha^2| = 1$ with $\alpha \notin \mathbb{Z}$. As $(\operatorname{tr} M)^2 < 4 \det M = 4$, we have $\operatorname{tr} M \in \{0, \pm 1\}$, since it must be an integer. For $\operatorname{tr} M = 0$, we see that α satisfies $\alpha^2 + 1 = 0$, Hence $\alpha = \pm i$. For $\operatorname{tr} M = -1$, α satisfies

$$\alpha^2 + \alpha + 1 = 0 \implies \alpha^3 = 1,$$

where α is then a 3-rd root of unity, and for $\operatorname{tr} M = 1$, we have

$$\alpha^2 - \alpha + 1 = 0 \implies \alpha^3 + 1 = 0 \implies \alpha^6 = 1,$$

where it is easily seen that α must be a primite 6-th root of unity.

For $\alpha = \pm i$, we may assume $\alpha = i$, by possibly composing multiplication by α with $z \mapsto -z$. Hence multiplication by α is a $\pi/2$ rotation of Λ, and

$i \in \Lambda$. This is realizable by a square lattice, generated by $\{1, i\}$. For α the 6-th root of unity, this is realizable by an hexagonal lattice, generated by $\{1, \alpha\}$.

Now if $f(z) = z + c$ for $c = a + b\tau$, If $a, b \in \mathbb{Q}$, then for some $N \in \mathbb{N}$, we have that $f^N(z) = z + Nc = z + Na + Nb\tau$ where $Na, Nb \in \mathbb{Z}$, so that f^N is the identity. Hence every orbit is periodic for f, and consists of a finite set.

Assume $b \notin \mathbb{Q}$ but $a \in \mathbb{Q}$. Then for some iterate $f^N(z) = z + Nc = z + Nb\tau$, and by the irrationality of b, the orbit of f^N will be dense on the "side" corresponding to τ. We have a finite set of circles corresponding to the translates of this side by the multiples of a. The analogous situation happens when $a \notin \mathbb{Q}$ but $b \in \mathbb{Q}$.

Now if both $a, b \notin \mathbb{Q}$, we will have density of the orbit on $\mathbb{T}$. (How to prove? Argument by pigeonhole principle, likely.) □

An observation is that, if $\alpha \neq 1$, the map $f(z) = \alpha z + c$ is conjugate to the map $g(z) = \alpha z$. For the choices of $\alpha \notin \mathbb{Z}$ and $|\alpha|^2 = 1$, can we prove that the lattices must necessarily be the square and hexagonal ones?

Problem (6-c. Grand orbit finite points)**.** Show that a nonlinear holomorphic map $f : \mathbb{C} \to \mathbb{C}$ has at most one grand orbit finite point. Show by examples such as $f(z) = \lambda z e^z$ and $f(z) = z^2 e^z$ that this fixed point need not be attracting, and in fact can have arbitrary multiplier.

Proof. We apply the strong Picard Theorem: if $f : \mathbb{C} \to \mathbb{C}$ is not a polynomial, having an essential singularity at infinity, for any neighborhood $\mathbb{C} \setminus \overline{\mathbb{D}_R}$ of infinity, the image $f(\mathbb{C} \setminus \overline{\mathbb{D}}_R)$ omits at most one point of $\mathbb{C}$.

The case for f a polynomial is already contemplated on $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Suppose there were at least two grand orbit finite points, and let S be the finite set of their grand orbits. Then $|S| > 1$ and S is contained in some ball $\mathbb{D}_R$. But by the strong Picard theorem, S has a preimage outside of it, contradicting that it is invariant under f.

For $f(z) = \lambda z e^z$, we see that for $\lambda \neq 0$, $f(z) = 0 \iff z = 0$, so 0 is grand orbit finite. Moreover, $f'(z) = \lambda(z+1)e^z$, so $f'(0) = \lambda$, where $\lambda \in \mathbb{C}^*$. So its multiplier is arbitrarily nonzero. For $f(z) = z^2 e^z$, it is easy to see that 0 is the unique point with finite grand orbit, and it is a critical point for f. □

7 Smooth Julia Sets

Problem (7-b)**.** For any $a \in \mathbb{D}$ the map

$$\phi_a(z) = \frac{z - a}{1 - \overline{a} z}$$

carries the unit disk isomorphically onto itself. A finite product of the form

$$f(z) = e^{i\theta}\phi_{a_1}(z)\dots\phi_{a_n}(z)$$

with $a_j \in \mathbb{D}$ is called a *Blascke product* of degree n. Show that every such f is a rational map which carries $\mathbb{D}$ onto $\mathbb{D}$ and $\hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ onto $\hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$. Conclude that the Julia set $J(f)$ is contained in the unit circle. If $g(z) = 1/f(z)$, interchanging the interior and exterior of the unit circle, show that $J(g)$ is also contained in the unit circle. If $n \geq 2$, and if one of the factors is $\varphi_0(z) = z$ show that f has attracting fixed points at zero and infinity, and show that $J(f)$ is the entire unit circle.

Proof. It is standard to see that ϕ_a is a rational map of degree 1, hence an automorphism of $\hat{\mathbb{C}}$, and

$$\left|\frac{z-a}{1-\overline{a}z}\right|^2 - 1 = \frac{-1}{|1-\overline{a}z|^2}(1-|z|^2)(1-|a|^2),$$

so that $|\phi_a(z)| < 1 \iff |z| < 1$, $|\phi_a(z)| > 1 \iff |z| > 1$ and ϕ_a preserves the unit circle.

This is sufficient to show that f is a rational map, and maps $\mathbb{D}$ to $\mathbb{D}$, S^1 to S^1, and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ to $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Note that $\phi_a \circ \phi_{-a} = \mathrm{Id}$. If f is non-constant, as it is a rational map, it must be surjective, so f maps $\mathbb{D}$ onto iteself, and analogously for S^1 and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

As $\mathbb{D}$ and $\hat{\mathbb{C}}\backslash\overline{\mathbb{D}} \cong \mathbb{D}$ are hyperbolic Riemann surfaces, families of holomorphic endomorphisms on them are always normal, hence $\mathbb{D} \cup (\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}) \subseteq F(f)$, and therefore $J(f) \subseteq S^1$. The same reasoning applies to $g = 1/f$, since $\mathbb{D}$ maps onto $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ maps onto $\mathbb{D}$. If $n \geq 2$ and $\phi_0(z)$ is one of the factors, Then we see that 0 is a fixed point for f. Moreover, $f(z) = zh(z)$ where h are the other factors, so

$$f'(z) = h(z) + zh'(z) \implies f'(0) = h(0),$$

and we know that $|f'(0)| = |h(0)| < 1$. Hence 0 is an atracting fixed point for 0. By a change of coordinates, it also easy to see that ∞ must be an attracting fixed point for f. By connectedness of components and the fact that boundaries of attracting basins are in the Julia set, we have that $\mathbb{D} \subseteq \mathcal{A}(0)$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \subseteq \mathcal{A}(\infty)$. By the jump discontinuity at S^1, we must have that $S^1 \subseteq J(f)$, hence all inclusions are equalities. ☐

Problem (7-c. Chebyshev Polynomials)**.** Define monic polynomials

$$P_1(z) = z, \quad P_2(z) = z^2 - 2, \quad P_3(z) = z^3 - 3z, \ \dots$$

inductively by the formula $P_{n+1}(z) + P_{n-1}(z) = zP_n(z)$. Show that $P_n(w + w^{-1}) = w^n + w^{-n}$, or equivalently $P_n(2\cos\theta) = 2\cos(n\theta)$, and show that $P_m \circ P_n = P_{mn}$. For $n \geq 2$ show that the Julia set of $\pm P_n$ is the interval $[-2, 2]$. For $n \geq 3$ show that P_n has $n - 1$ distinct crtical points but only two critical values, namely ± 2.

Proof. Suppose by induction that $P_k(w + w^{-1}) = w^k + w^{-k}$ for $1 \leq k \leq n$, which is evidently true for $k = 1$ and true for $k = 2$, as

$$(w + w^{-1})^2 - 2 = w^2 + 2ww^{-1} + w^{-2} - 2 = w^2 + w^{-2}.$$

Then

$$\begin{aligned} P_{n+1}(w + w^{-1}) &= (w + w^{-1})P_n(w + w^{-1}) - P_{n-1}(w + w^{-1}) \\ &= (w + w^{-1})(w^n + w^{-n}) - w^{n-1} - w^{-(n-1)} \\ &= w^{n+1} + w^{n-1} + w^{-n+1} + w^{-n-1} - w^{n-1} - w^{1-n} \\ &= w^{n+1} + w^{-(n+1)}, \end{aligned}$$

which completes the induction step. Moreover, as $2\cos\theta = e^{i\theta} + e^{-i\theta}$, we have the second equivalence.

Note that

$$P_m(P_n(w+w^{-1})) = P_m(w^n+w^{-n}) = (w^n)^m+(w^{-n})^m = w^{mn}+w^{-mn} = P_{mn}(w+w^{-1}),$$

And as this equality holds for infinitely many values of $w + w^{-1}$, we have that the polynomials $P_m \circ P_n$ and P_{mn} are equal.

For $n \geq 2$, consider the map $f(z) = z^n$ on $\hat{\mathbb{C}}$, and $h(w) = w + w^{-1}$. We then have that $P_n \circ h = h \circ f$, where then we have a holomorphic semiconjugacy:

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \hat{\mathbb{C}} & \xrightarrow{P_n} & \hat{\mathbb{C}} \end{array}$$

We have previously seen that h maps the exterior of the unit disk isomorphically onto $\hat{\mathbb{C}} \setminus [-2, 2]$. The dynamics are conjugate on these sets, so that for a point outside $[-2, 2]$ it will diverge to infinity. Hence $J(P_n) \subseteq [-2, 2]$. For another approach, one sees that $[-2, 2]$ is fully invariant under P_n, containing the repelling fixed point $2 = 2\cos 0$, so that $J(P_n) \subseteq [-2, 2]$.

Now we must have that $\partial\mathcal{A}(\infty) \subseteq J(P_n) \subseteq [-2, 2]$. If U were another Fatou component of P_n, it would have to be bounded and $\partial \subseteq [-2, 2]$. But $ImP_n(z)$ must attain a maximum at the boundary of U, so it is constantly equal to zero, giving a contradiction. Hence there are no other Fatou components, and $\mathcal{A}(\infty) = \hat{\mathbb{C}} \setminus [-2, 2]$, and $J(P_n) = [-2, 2]$.

We also see that, since $P_n(2\cos\theta) = 2\cos(n\theta)$, by taking the derivative,

$$\begin{aligned} -2\sin\theta P_n'(2\cos\theta) &= -2n\sin(n\theta) \\ \implies P_n'(2\cos\theta) &= n\frac{\sin(n\theta)}{\sin\theta}, \end{aligned}$$

which is zero when $\sin(n\theta) = 0$ but $\sin\theta \neq 0$, so $\theta = \pi k/n$ for $k \in \mathbb{Z} \setminus n\mathbb{Z}$. These yield $n-1$ distinct values of $2\cos\theta$, corresponding to the $n-1$ roots of the polynomial P_n'. In all of the cases, $\cos(n\theta) = \pm 1$, yielding the the critical values ± 2. □

Problem (7-d. More interval Julia sets)**.** Now suppose that f is a Blaschke product with real coefficients, and with an attracting fixed point at the origin. Show that there is one and only one rational map F of the same degree so that the following diagram is commutiative:

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{\;f\;} & \hat{\mathbb{C}} \\ {\scriptstyle z+1/z}\downarrow & & \downarrow{\scriptstyle z+1/z} \\ \hat{\mathbb{C}} & \xrightarrow{\;F\;} & \hat{\mathbb{C}} \end{array}$$

and show that $J(F) = [-2, 2]$. In the special case $f(z) = z^n$, show that this construction yields the Chebyshev polynomials.

Proof. Recall that the map $h(z) = z + 1/z$ maps the unit circle two to one onto the interval $[-2, 2]$, and is a conformal isomorphism from $\mathbb{D}$ (and from $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$) onto $\hat{\mathbb{C}} \setminus [-2, 2]$.

In order for F to satisfy $F(h(z)) = h(f(z))$, for $z \in \mathbb{D}$, we must have that

$$F(w) = h(f(z)),$$

where z is either of the two preimages of $w \in \hat{\mathbb{C}}$. There are exactly two solutions of $w = z + z^{-1}$, one being the inverse of the other, so as a necessary condition we must have that $h(f(z)) = h(f(z^{-1}))$ for all z. This happens when either $f(z) = f(z^{-1})$ or $f(z)^{-1} = f(z^{-1})$; as f is a Blaschke product, it maps $\mathbb{D}$ onto $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, hence we must have that $f(z)^{-1} = f(z^{-1})$.

Note that when $a \in \mathbb{R} \cap \mathbb{D}$, we have that

$$\phi_a(z^{-1}) = \frac{1/z - a}{1 - a/z} = \frac{1 - az}{z - a} = \phi_a(z)^{-1},$$

so that indeed f will satisfy the latter necessary condition $f(z^{-1}) = f(z)^{-1}$.

Treating $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as a quotient map, we see that $h \circ f$ passes continuously to the quotient as F, and on $\hat{\mathbb{C}} \setminus [-2, 2]$, it is defined holomorphically as $F(w) = f(h(z))$, where z is the solution in $\mathbb{D}$, say. Note that h has critical points only at ± 1, so outside of them, by taking neighborhoods on which h is invertible, F is defined holomorphically. By Riemann's removable singularity theorem, F extends holomorphically to ± 1.

As an endomorphism of the Riemann sphere, F is a rational function, and has been uniquely described by the necessary conditions imposed. It must have degree equal to the degree of f, as, since the dynamics are conjugate from $\mathbb{D}$ to $\hat{\mathbb{C}} \setminus [-2, 2]$ and a generic point on $\mathbb{D}$ has number of preimages by f equal to the degree, the same will happen for F. It is also easy to see that for $f(z) = z^n$, the situation thus described is the same as in the previous problem.

Finally, by the same previous arguments, we may conclude that $J(F) = [-2, 2]$, since $J(f) = S^1$ and all point in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ converge to infinity. □

Problem (7-g. The family of degree four Lattès maps)**.** For the torus $\mathbb{T} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, show that the involution $z \mapsto z + 1/2$ of $\mathbb{T}$ corresponds under $\wp$ to an involution of the form $w \mapsto a/w$ of $\hat{\mathbb{C}}$, with fixed points at $w = \pm\sqrt{a}$. Show that the rational map $f = f_a$ has poles at $\infty, 0, 1, a$ and double zeros at $\pm\sqrt{a}$. Show that f has a fixed point of multiplier $\lambda = 4$ at infinity, and conclude that

$$f(w) = \frac{(w^2 - a)^2}{4w(w-1)(w-a)}.$$

As an example, if $a = -1$ then

$$f(w) = \frac{(w^2 + 1)^2}{4w(w^2 - 1)}.$$

Show that the correspondence $\tau \mapsto a = a(\tau) \in \mathbb{C} \setminus \{0, 1\}$ satisfies the equations

$$a(\tau + 1) = 1/a(\tau), \quad a(-1/\tau) = 1 - a(\tau), \quad a(-\overline{\tau}) = \overline{a(\tau)}.$$

Conclude, for example, that $a(i) = 1/2$, and that $a(\frac{1+i}{2}) = -1$.

Proof. We see that the involution $z \mapsto z + 1/2$ commutes with multiplication by -1, as

$$(-z) + 1/2 \sim -z - 1/2 = -(z + 1/2),$$

hence it descends to a continuous map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $g(\wp(z)) = \wp(z + 1/2)$. It has to be holomorphic where away from the critical values of

$\wp$, and since they are isolated, by Riemann's removable singularity theorem it extends (uniquely) to a rational map on $\hat{\mathbb{C}}$. Moreover, its degree must be 1, given that the degree of the involution also is 1.

We see in fact that

$$g(g(\wp(z)) = g(\wp(z + 1/2)) = \wp(z + 1) = \wp(z),$$

so that $g \circ g = \mathrm{Id}$ on $\hat{\mathbb{C}}$. Since

$$(1/2 + \tau/2) + 1/2 \sim \tau/2$$

We see that the four points

$$\frac{1}{4}, \quad \frac{3}{4}, \quad \frac{1}{4} + \frac{1}{2}\tau \quad \text{and } \frac{3}{4} + \frac{1}{2}\tau$$

on the torus are the solutions to $z+1/2 \sim -z$, so that they map to two fixed points of g on $\hat{\mathbb{C}}$. Recall that by our construction of $\wp = \wp_\tau$, we have

$$\wp(0) = \infty, \quad \wp\left(\frac{1}{2}\right) = 0, \quad \wp\left(\frac{\tau}{2}\right) = 1, \quad \wp\left(\frac{1+\tau}{2}\right) = a,$$

so that $f(\wp(z)) = \wp(2z)$. Hence

$$g(0) = \infty, \quad g(\infty) = 0, \quad g(1) = a, \quad g(a) = 1.$$

By considering the map $h(w) = 1/g(w)$ which fixes infinity and 0, hence $h(w) = bw$ for some b, we have that $b = 1/a$. In conclusion,

$$g(w) = \frac{a}{w}.$$

This has fixed points at $\pm\sqrt{a}$, the images of the four points on the torus previously mentioned.

Note that the doubling map on $\mathbb{T}$ satisfies

$$2 \cdot 0 = 2 \cdot \frac{1}{2} = 2 \cdot \frac{\tau}{2} = 2 \cdot \frac{1+\tau}{2} = 0,$$

and pushing this down to $f = f_a$, we have

$$f(\infty) = f(0) = f(1) = f(a) = \infty.$$

These must all be non-critical points of f. Note also that

$$2 \cdot \frac{1}{4} = 2 \cdot \frac{3}{4} = \frac{1}{2}, \quad 2 \cdot \left(\frac{1}{4} + \frac{\tau}{2}\right) = 2 \cdot \left(\frac{3}{4} + \frac{\tau}{2}\right) = \frac{1}{2},$$

and pushing this down by $\wp$ to f we have only two distinct preimages of $\wp(1/2) = 0$. Hence $\pm\sqrt{a}$ are two simple critical points, being double zeroes of f.

Note that 0 is a fixed point of the doubling map on $\mathbb{T}$, pushing down to $S = \hat{\mathbb{C}}$ to the fixed point at ∞. Moreover, given that the multiplier of 0 for $z \mapsto 2z$ is 2, and a uniformizing neighborhood of 0 in the quotient $S = \mathbb{T}/(z \sim -z)$ is z^2, the multiplier of 0 on S becomes 4, and this is the multiplier of ∞ for f.

Therefore the map

$$f(w) = \frac{w(w-1)(w-a)}{(w^2-a)^2}$$

has no zeros and no poles, hence it must be constant, so that

$$f(w) = \lambda \frac{(w^2-a)^2}{w(w-1)(w-a)}.$$

By calculating $f'(w)$ and $\lim_{n\to\infty} 1/f'(w) = 4$, we have that $\lambda = 1/4$, and

$$f_a(w) = \frac{(w^2-a)^2}{4w(w-1)(w-a)}.$$

Now, given the matrix

$$M = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

M acts on the upper half plane $\mathbb{H}$ by $\tau \mapsto (m\tau+n)/(p\tau+q)$. Note that the lattice $\mathbb{Z} + M\tau\mathbb{Z}$ is equivalent (through rescaling by a complex number) to the lattice $(p\tau+q)\mathbb{Z} + (m\tau+n)\mathbb{Z}$, which is equal to $\mathbb{Z}+\tau\mathbb{Z}$. We in fact have

$$\begin{aligned} xM\tau + y \mapsto\ & x(m\tau+n) + y(p\tau+q) \\ &= (mx+py)\tau + (nx+qy), \end{aligned}$$

which is represented by the linear transformation

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} m & n \\ p & q \end{pmatrix},$$

with respect to the bases $\{M\tau, 1\}$ and $\{\tau, 1\}$.

This map $\mathbb{Z} + M\tau\mathbb{Z} \to \mathbb{Z} + \tau\mathbb{Z}$ given by $z \mapsto (p\tau+q)z$ descends to a conformal isomorphism

$$\mathbb{C}/(Z + M\tau\mathbb{Z}) \xrightarrow{h} \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}),$$

which preserves the linear structure. Composing this with the map $x\tau + y \mapsto xM\tau + y$, we in fact obtain the automorphism on $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ given by the matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ acting as a linear transformation.

Given $\wp_\tau : \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \to \hat{\mathbb{C}}$ and $\wp_{M\tau} : \mathbb{C}/(\mathbb{Z} + M\tau\mathbb{Z}) \to \hat{\mathbb{C}}$, we wish to compare $\wp_\tau$ and $\wp_{M\tau} \circ h^{-1}$. Both of them will be double branched covers of $\hat{\mathbb{C}}$ ramified at $0, 1/2, \tau/2$ and $(1 + \tau)/2$, hence will be related by a Möbius transformation. Recall that $\wp_{M\tau}$ is defined by

$$\wp_{M\tau}(0) = \infty, \quad \wp_{M\tau}\left(\frac{1}{2}\right) = 0, \quad \wp_{M\tau}\left(\frac{M\tau}{2}\right) = 1, \quad \wp_{M\tau}\left(\frac{1 + M\tau}{2}\right) = a(M\tau).$$

As the isomorphism h preserves the linear structure of the torii, it must map 0 to 0 and send

$$\{1/2, M\tau/2, (1 + M\tau)/2\} \mapsto \{1/2, \tau/2, (1 + \tau)/2\},$$

and from the above discussion, we in fact have that $\wp_{M\tau} \circ h^{-1} = \wp_\tau \circ M$. (?) We therefore only need to see how the automorphism M permutes the ramification points of the torii to compare $\wp_\tau$ and $\wp_{M\tau}$.

For $M\tau = \tau + 1$, we have that

$$\begin{aligned} &\{0, 1/2, \tau/2, (1 + \tau)/2\} \longmapsto \{0, 1/2, (1 + \tau)/2, \tau/2\} \\ &\implies \{\infty, 0, 1, a(\tau)\} \longmapsto \{\infty, 0, a(\tau + 1), 1\}, \end{aligned}$$

so that $a(\tau + 1) = 1/a(\tau)$. For $M\tau = -1/\tau$, we have

$$\begin{aligned} &\{0, 1/2, \tau/2, (1 + \tau)/2\} \longmapsto \{0, \tau/2, 1/2, (1 + \tau)/2\} \\ &\implies \{\infty, 0, 1, a(\tau)\} \longmapsto \{\infty, 1, 0, a(-1/\tau)\}, \end{aligned}$$

so that $a(-1/\tau) = 1 - a(\tau)$.

For $-\overline{\tau}$, we have a reflection of the lattice. In handwavy terms, this will induce a conformal structure on $\hat{\mathbb{C}}$ for which the identity is an anticonformal map, so that conjugation will be a conformal isomorphism, and $(-\overline{\tau}) = \overline{a(\tau)}$. (?) □

8 Geometrically Attracting or Repelling Fixed Points

Comment on the Proof of 8.5:

Recall the following from the book. From Köenigs linearization, if $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational function of degree $d \geq 2$ and $\hat{z}$ is a geometrically attracting fixed point with basin of attraction $\mathcal{A} \subset \hat{\mathbb{C}}$, we find a holomorphic map $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(\hat{z}) = 0$ and

$$\lambda\phi(z) = \phi(f(z))$$

for all $z \in \mathcal{A}$, and $\phi'(0) = 1$, being a conformal isomorphism of some neighborhood of $\hat{z}$ to some disk $\mathbb{D}_\varepsilon(0)$ around 0. If $\mathcal{A}_0$ is the immediate basin of attraction of $\hat{z}$, we have a locally defined holomorphic inverse $\psi_\varepsilon : \mathbb{D}_\varepsilon \to \mathcal{A}_0 \subset \hat{\mathbb{C}}$, such that $\psi_\varepsilon(0) = \hat{z}$.

The book claims that it is not possible to exetend ψ_ε by analytic continuation along radial lines to all of $\mathbb{C}$, as that would yield a holomorphic map $\psi : \mathbb{C} \to \mathcal{A}_0 \subset \hat{\mathbb{C}}$ satisfying $\phi(\psi(w)) = w$. Then ψ is injective and ϕ is surjective.

The book claims this is possible only if $\hat{\mathbb{C}} \setminus \psi(\mathbb{C})$ consisted of a single point. We can see this more clearly through the fact that ψ will have to be a conformal isomorphism, so that $\psi(\mathbb{C})$ is a copy of $\mathbb{C}$ as a simply connected euclidean Riemann surface inside $\hat{\mathbb{C}}$.

This yields a contradiction also in the sense that then $J(f) \subseteq \hat{\mathbb{C}} \setminus \psi(\mathbb{C})$ would be finite. The book claims that $f|_{\psi(\mathbb{C})}$ is one-to-one: for $w, w \in \mathbb{C}$,

$$\begin{aligned} f(\psi(w)) = f(\psi(w')) &\implies \phi(f(\psi(w))) = \phi(f(\psi(w'))) \\ &\implies \lambda\phi(\psi(w)) = \lambda\phi(\psi(w')) \implies w = w'. \end{aligned}$$

That ψ is a left inverse of ϕ, due to being an extension of a locally defined inverse, is derived from the following:

Lemma 8.1. *Suppose $g : S \to T$ is a holomorphic function mapping p to q where p is not a critical point, and $h : U \to S$ is a local inverse of f such that $h(q) = p$, where U can be taken open and connected.*

If $V \subseteq T$ is open, connected and contains U, and $H : V \to S$ is a holomorphic function which extends h, then $g \circ H = \mathrm{Id}_U$.

The above is true by the identity principle: $g \circ H$ agrees with $g \circ h = \mathrm{Id}$ on a smaller open set, and since V is connected, we must have that $g \circ H = \mathrm{Id}$. In particular, H is univalent, so that $H : V \to H(V)$ is a conformal isomorphism, and $g|_{H(V)} = H^{-1}$ is also a conformal isomorphism.

I believe the same principle will hold true for analyitic continuation of the local inverse h along curves in T, where, by the identity principle, the germs $\widetilde{h}$ along the analytic continuation will always have to satisfy $g \circ \widetilde{h} = \mathrm{Id}$ on a neighborhood of the germ, whenever the analytic continuation is defined. (I trust that analytic continuation is well defined for germs of holomorphic functions with values in another Riemann surface S, not necessarily $\mathbb{C}$.)

In essence, the book takes a maximal analytic continuation of the local inverse ψ_ε to a disk $\mathbb{D}_r$, where then $\phi \circ \psi = \text{Id}$, $\psi : \mathbb{D}_r \to \mathcal{A}_0$ is univalent, $\phi|_{\psi(\mathbb{D}_r)} : \psi(\mathbb{D}_r) \subset \mathcal{A}_0 \to \mathbb{D}_r$ is a conformal isomorphism inverse to ψ and f is one-to-one on $\psi(\mathbb{D}_r)$.

Another comment: later in the proof, we want to show that $\phi : \overline{U} \to \overline{\mathbb{D}_r}$ is a homeomorphism, and to this end, the book claims it is sufficient to show $\phi : \partial U \to \mathbb{C}$ is injective where naturally ∂U maps to $\overline{\mathbb{D}_r}$. The fact that ∂U must map to $\partial \mathbb{D}_r$ comes from the properness of $\phi|_U : U \to \mathbb{D}_r$, being a conformal isomorphism. If $\phi|_{\partial U}$ is injective, then $\phi|_{\overline{U}}$ is an injective map on a compact set (as it is ultimately contained in $\hat{\mathbb{C}}$!), Hence a homeomorphism onto its image. Since it must be compact and contains $\mathbb{D}_r$, it must contain $\overline{\mathbb{D}_r}$. Hence the image is exactly $\overline{\mathbb{D}_r}$.

Comment on the proof of 8.7:
To be more explicit and concrete about the proof, suppose $k = 1$ (by possibly taking the iterate f^k). We have $\overline{N_i} \subseteq N_{i+1}$ compactly contained, and assume N_1 has $\Gamma_1, \ldots, \Gamma_m$ boundary curves for $m \geq 2$. Naturally $\hat{\mathbb{C}} \setminus \overline{N_1}$ has exactly m connected components, for if C is a component of $\hat{\mathbb{C}} \setminus \overline{N_1}$, then ∂C must intersect $\partial N_1 = \bigsqcup_i \Gamma_i$, and in fact be equal to one of the curves Γ_i.

Moreover, $N_2 \setminus \overline{N_1}$ must have at least m components. This is because we have the inclusion

$$N_2 \setminus \overline{N_1} \subseteq \hat{\mathbb{C}} \setminus \overline{N_1},$$

and it is surjective on connected components. That is, for every component V of $\hat{\mathbb{C}} \setminus \overline{N_1}$, there is at least one component U of $N_2 \setminus \overline{N_1}$ such that $U \subseteq V$. Notably U must be the component whose boundary is the corresponding curve Γ_i, since each Γ_i is contained in N_2.

For each component U of $N_2 \setminus \overline{N_1}$, the map $f : U \to N_1 \setminus \overline{N_0}$ is a branched covering, so that U has at least $m+1$ boundary curves, m of which correspond to the $\Gamma_1, \ldots, \Gamma_m$ and at least one which will correspond to ∂N_0.

If V is a component of $\hat{\mathbb{C}} \setminus N_1$, with $\partial V = \Gamma_i$ for some i, we have a component U of $N_2 \setminus \overline{N_1}$ contained in V with one of its boundary curves being the Γ_i, that is, bounding N_1. Also note that this Γ_i corresponds to a preimage of the curve ∂N_0. Consider its other m bondary curves, more specifically the preimages of the $\Gamma_1, \ldots, \Gamma_m$; they will corresponds to at least m distinct components of $\hat{\mathbb{C}} \setminus N_2$. This concludes the reasoning of the proof.

Problem (8-b. Global linearization)**.** Suppose that $f : S \to S$ is a holomorphic map from the Riemann surface S to itself. (For example suppose $S = \hat{\mathbb{C}}$, so that f is a rational map.) Show that the linearizing map ϕ maps the attracting basin $\mathcal{A}$ onto $\mathbb{C}$. Show that $p_0 \in \mathcal{A}$ is a critical point of ϕ if and only if the orbit $p_n = f^n(p_0)$ contains some critical point of f.

Proof. Recall that $\phi : \mathcal{A} \to \mathbb{C}$ is the holomorphic map such that

$$\phi(f(z)) = \lambda\phi(z),$$

for all $z \in \mathcal{A}$. We know that some disk $\mathbb{D}_\varepsilon$ is contained in the image of ϕ. As $f : S \to S$ is surjective and $\mathcal{A}$ is totally invariant, $f|_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ is surjective. If $m_\lambda : \mathbb{C} \to \mathbb{C}$ is the multiplication by λ and $\phi|_U :\to V$ is surjective, then ϕ maps $f^{-1}(U)$ surjectively onto $\lambda^{-1}V$ by the conjugation. Explicitly, if $w = \phi(z) \in V$ and z' is such that $f(z') = z$, which always exists, then $\phi(z) = \lambda\phi(z')$, so $\phi(z') = \lambda^{-1}\phi(z)$.

As the image of ϕ contains a disk $\mathbb{D}_\varepsilon$, it will contains all disks $\mathbb{D}_{\lambda^{-n}\varepsilon}$, and hence ϕ is surjective onto $\mathbb{C}$.

If $p_0 \in \mathcal{A}$, recall that $\phi(p_0)$ is defined as $\phi(f^n(p_0))/\lambda^n$ for any $n \in \mathbb{N}$ such that $\phi : U \to \mathbb{D}_\varepsilon$ is defined biholomorphically on a neighborhood of the geometrically attracting fixed point. The same formula will also hold true for points in a neighborhood of p_0. Hence

$$\phi'(p_0) = \lambda^{-n}\phi'(f^n(p_0))(f^n)'(p_0) = \lambda^{-n}\phi'(f^n(p_0))\prod_{k=0}^{n-1} f'(f^k(p_0)).$$

as ϕ is biholomorphic on a neighborhood of the fixed point, where we may assume f also does not have critical points, $\phi'(z_0)$ is zero if and only if some positive iterate of p_0 is a critical point for f. If $f^m(p_0)$ is a critical point, we may take inthe above $n > m$ such that the expression is still well defined and have p_0 a critical point of ϕ. $\square$

Problem (Asymptotic values)**.** In order to extend theorem 8.6 to a non-compact Riemann surface such as $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$, we need some definitions. Let $f : S \to S'$ be a holomorphic map between Riemann surfaces. A point $v \in S'$ is a *critical value* if it is the image under f of a *critical point*, that is a point at which the first derivative of f vanishes. It is an *asymptotic value* if there exists a continuous path $p : [0, 1) \to S$ which "diverges to infinity" in S, or in other words eventually leaves any compact subset of S, but whose image under f converges to the point v. A connected open set $U \subset S'$ is *evenly covered* if every component of $f^{-1}(U)$ maps homeomorphically onto U, and that f is a covering map if every neighborhood of S' has a neighborhood which is evenly covered.

Show that a simply connected open subset of S' is evenly covered by f if and only if it contains no crtical value or asymptotic values. In particular, f is a covering map if and only if S' contains no critical values and no asymptotic values.

For a holomorphic self-map $f : S \to S$, show that the immediate basin of any attracting periodic orbit must contain either a critical values or an asymptotic value or both, except in the special case of a linear map from $\mathbb{C}$ or $\hat{\mathbb{C}}$ to itself. As an example, for any $c \neq 0$, show that the transcendental map $f(z) = ce^z$ from $\mathbb{C}$ to itself has no critical points, and just one asymptotic value, namely $z = 0$. Conclude that it has at most one periodic attractor. If $|c| < 1/e$ show that f maps the unit disk into itself, and that f has an attracting fixed point in this disk.

The map f is *proper* if the pre-image $f^{-1}(K)$ of every compact set $K \subset S'$ is a compact subset of S. Show that a proper map has no asymptotic values.

Proof. Let $U \subseteq S'$ be non-empty, open and simply connected (in particular also connected). We know that at any critical point p of f, we may find holomorphic charts around p and $f(p)$ such that locally f is of the form $z \mapsto z^n$, for some $n \geq 2$. This implies that f is not locally one-to-one at a critical point, so that if U contained a critical value, in the connected component of its preimage that contains the critical point, f cannot be a homeomorphism.

Suppose that $v \in U$ is an asymptotic value and $p : [0, 1) \to S$ is a path in S that diverges to infinity and

$$\lim_{t\to 1} f(p(t)) = v.$$

We may further assume that $f(p[0, 1)) \subset U$. Let $\hat{U}$ be the component of the preimage $f^{-1}(U)$ containing the path, as $\hat{U}$ is connected. $f : \hat{U} \to U$ is a homeomorphism, and in particular proper. If K is a compact neighborhood of v, then the path p eventually leaves $f^{-1}(K)$, hence the image of the path could not converge to f. Hence if U is evenly covered, it cannot contain asymptotic values as well.

Conversely, assume U contains no critical and asymptotic values, and $\hat{U}$ is a connected component of $f^{-1}(U)$. Then $\hat{f} = f|_{\hat{U}}$ is surjective, a local homeomorphism, and is proper, since U has no asymptotic values. For if $K \subseteq U$ were compact and $f^{-1}(K)$ not, there exists a sequence $(z_n)_n \subset f^{-1}(K)$ that has no convergent subsequence, hence leaves every compact set of S, but $\lim_{n\to\infty} f(z_n) = v \in K$.

If we admit the theory of ends, we may take a subsequence of the z_n that converges to an end e of S, and connected the points of the sequence by a continuous path that lies in smaller and smaller neighborhoods of this end. This will show that v is an asymptotic value, a contradiction.

A proper local homeomoprhism is a covering map, and since U is simply connected, this covering map must be a homeomorphism. Hence U is evenly covered. The above reasonings are repeated to show that f is a covering map if and only if S' contains no critical values and no asymptotic values.

Now let p be an attracting fixed point of $f : S \to S$. We may assume p is geometrically attracting but not superattracting. By Koenigs linearization, we still have the map $\phi : \mathcal{A} \to \mathbb{C}$ that conjugates f with the multiplication $w \mapsto \lambda w$, and a locally defined inverse $\psi_\varepsilon : \mathbb{D}_\varepsilon \to \mathcal{A}$, with maximal extension $\psi : \mathbb{D}_r \to \mathcal{A}$ (by analytic continuation along radial lines?) such that $\phi \circ \psi =$ Id. (Possibly $r = \infty$.)

Hence ψ is univalent on $\mathbb{D}_r$, being a conformal isomorphism onto its image $\psi(\mathbb{D}_r)$. Then $\phi|_{\psi(\mathbb{D}_r)} = \psi^{-1}$ is also a conformal isomorphism onto $\mathbb{D}_r$, and f is one-to-one in $\psi(\mathbb{D}_r)$.

If ψ were defined on the whole plane $\mathbb{C}$, then $\mathcal{A} = \mathbb{C}$ or $\hat{\mathbb{C}}$, and the same holds for S. As f is one-to-one on $\psi(\mathbb{C})$, the only possible case for that is a linear map on $\mathbb{C}$ or $\hat{\mathbb{C}}$. In all other situations, ψ must be defined on some maximal disk $\mathbb{D}_r$ with $r < \infty$.

As in the proof of 8.5, if $U = \psi(\mathbb{D}_r)$, we have that $\overline{U}$ is contained in $\mathcal{A}$ (but is not necessarily compact!), because $\lambda\mathbb{D}_r$ is compactly contained in $\mathbb{D}_r$, so that $f(U)$ is compactly contained in U. If $\partial\mathbb{D}_r$ were to contain no critical values and no asymptotic values, the proof of 8.5 follows verbatim, but this time making explicit the fact that $\psi(tw_0)$, being a path in S such that whose limit under ϕ is w_0, cannot diverge to infinity, and therefore has an accumulation point z_0 as in the proof. Hence $\mathcal{A}$ must contain either a critical value, or asymptotic value, or both.

Evidently $f(z) = ce^z$ has no critical points as $f'(z) = ce^z \neq 0$. If $p : [0,1) \to \mathbb{C}$ is a path diverging to infinity, for $p(t) = x(t) + iy(t)$, we have

$$f(p(t)) = ce^x e^{iy} = ce^{x(t)}(\cos y(t) + i \sin y(t)),$$

so that if $x(t)$ does not converge to $-\infty$, we must have that $x(t)$ and $y(t)$ must both converge, otherwise either the argument of the modulus would change wildly as $p \to \infty$. This shows that the only asymptotic vale can happen when $x(t) \to -\infty$, and the asymptotic value is 0. Therefore f can have at most one periodic attractor.

If $c < 1/e$, we have

$$|f(z)| = |c||e^z| = |c|e^x < e^{x-1} < e^{|z|-1},$$

so if $|z| < 1$, we have that $|f(z)| < 1$, so that $f(\mathbb{D}) \subseteq \mathbb{D}$. As f is not an automorphism of $\mathbb{D}$, by Shchwarz's lemma f must have an attracting fixed point in $\mathbb{D}$. This can also be seen by taking any $c < c' < 1/e$, where $|f(z)| < e^{(\log c')|z|}$, so that if $|z| < 1$, then $|f(z)| < c' < 1$ uniformly.

For the last question, we have already proved that if f is proper, then it can have no asymptotic values. □

Problem (The image $\psi(\mathbb{C}) \subset S$)**.** If $\hat{p}$ is a repelling point for the holomorphic map $f : S \to S$, show that the image of the map $\psi : \mathbb{C} \to S$ of 8.10 is everywhere dense, and in fact the complement of $S \setminus \psi(\mathbb{C})$ consists of grand orbit finite points.

Proof. Recall that $\psi : \mathbb{C} \to S$ is defined such that $\psi(0) = \hat{p}$ and

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\lambda\cdot} & \mathbb{C} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ S & \xrightarrow{f} & S \end{array}$$

and ψ is biholomorphic on a neighborhood of 0.

Using the same reasoning as a previous problem, as $\lambda\cdot$ is surjective, we have that $f^{-1}(\psi(\mathbb{C})) \subseteq \psi(\mathbb{C})$. For if $z \in \psi(\mathbb{C})$, so that $z = \psi(w)$, we have

$$f(\psi(\lambda^{-1}w)) = \psi(\lambda\lambda^{-1}w) = \psi(w) = z.$$

This implies that $\psi(\mathbb{C})$ is a backwards invariant open set intersecting the Julia set. If $S \setminus \psi(\mathbb{C})$ contained at least three points, then $\psi(C)$ would be hyperbolic. But then $\{f^n\}$ would be normal on $\psi(C)$, a contradiction with the fact that it intersects $J(f)$. Hence $S \setminus \psi(\mathbb{C})$ has at most two points, and is an invariant closed set. It must consist of grand orbit finite points, and $\psi(C)$ is dense in S. (in particular it is dense in $J(f)$.) □

Problem (8-f. Counting basin components)**.** Let $\mathcal{A}$ be the attracting basin of a periodic point which may either be superattracting or geometrically attracting. If some connected component of $\mathcal{A}$ is not periodic, show that $\mathcal{A}$ has infinitely many components. Suppose then that $\mathcal{A}$ has only finitely many components forming a periodic cycle. If these components are simply connected, use the Riemann-Hurwitz formula 7.2 to show that the period is at most two. (Example: $f(z) = 1/z^2$.) If they are infinitely connected, show that the period must be one.

Proof. Suppose U is a Fatou component of $\mathcal{A}$ such that U is not periodic (recall that we have already proved that the image of a Fatou component is also a Fatou component, if S is compact). In particular, U does not contain any point of the periodic orbit $\mathcal{O}$.

If $f|_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is surjective (which always occurs when S is compact), then we may take the Fatou components containing the components of $f^{-1}(U) \subseteq \mathcal{A}$. The iterated preimages of a component forming an orbit will have to be distinct, otherwise U would be periodic.

In fact, suppose $\hat{U}$ is a component of $f^{-1}(U)$, and let $\widetilde{U}$ be the Fatou component such that $\hat{U} \subseteq \widetilde{U}$. (Here we are still assuming that S is compact.)

By the same reasonings used for when there are no asymptotic values, we have that $f|_{\hat{U}} : \hat{U} \to U$ is proper, and also $f^{-1}(\partial U) \subseteq \partial \hat{U}$. But since this is contained in the Julia set, it must in fact be contained in $\partial \widetilde{U}$, and through some more handwaving we have that $\hat{U} = \widetilde{U}$.

In summary: when S is compact, the image of a Fatou component is a Fatou component, and any connected component of the preimage is also a Fatou component.

Now assume $\mathcal{A}$ consists of finitely many components, being exactly the ones containing the points of the periodic orbit $\mathcal{O}$ of period p.

If they are all simply connected, then $\chi(\mathcal{A}) = p$, and $f|_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is a branched cover. It is of degree d, since a generic point in $\mathcal{A}$ will have d preimages. If c is the number of critical points of f contained in $\mathcal{A}$, then by Riemann-Hurwitz:

$$c = d\chi(\mathcal{A}) - \chi(\mathcal{A}) = p(d-1).$$

But since $c \leq 2d-2$, we necessarily have that either $p = 1$ or $p = 2$.

If $p = 1$, then exactly $d-1$ critical points are contained in $\mathcal{A} = \mathcal{A}^0$. If $p = 2$, consisting of two components U_1 and U_2, considering the branched coverings $f : U_i \to U_{i+1}$, we have that exactly $d-1$ critical points are contained in U_1 and exactly $d-1$ critical points are contained in U_2.

If $\mathcal{A} = U_1 \cup \cdots \cup U_p$, assume now that at least one component U_i is infinitely connected (this can be assumed given the duality of simply connected/infinitely connected). By removing the postcritical set of U_{i+1} and its preimage in U_i, we get a covering map, and by injectiveness on the fundamental group, we have that U_{i+1} must also be infinitely connected. (Maybe this argument could be improved.) Hence all components of $\mathcal{A}$ are infinitely connected.

By conjugating f by a Möbius transformation, we may assume that one of the points in the periodic orbit is infinity, say U_p, so that all the other components are bounded components of $F(f)$. Let $K = \mathbb{C} \setminus U_p$, the compact set of all points whose orbit under f^p is bounded. Naturally K is totally invariant under f^p, and $U_1, \ldots, U_{p-1} \subseteq \operatorname{int} K$.

We know that f^p can have no poles in K, so that f^p is holomorphic on a neighborhood of K. By the maximum modulus principle, the maximum of $|f^p(z)|$ is attained on $\partial K = \partial U_p$, so that $\operatorname{int} K \subseteq F(f)$ by normality. But if U_i for $i = 1, \ldots, p-1$ were infinitely connected, there would be components of $J(f)$ in $\operatorname{int} K$, a contradiction. More explicitly, $\hat{\mathbb{C}} \setminus U_i$ has exactly one unbounded component, which contains $\overline{U_p}$. Therefore any other component must be contained in $\operatorname{int} K$. This concludes that $p = 1$.

□

9 Böttcher's Theorem and Polynomial Dynamics

Comment on Theorem 9.3:

If we try to reproduce the arguments for Köenigs linearization, where we have a homeomorphism between $\overline{U}$ and a disk in the $\mathbb{C}$ plane, we run into trouble. This is mainly because we are not given the function $\phi : \mathcal{A} \to \mathbb{C}$ "globally" like in the Köenigs case, so there is no precise map from the closure $\overline{U} \subseteq \mathcal{A}$ that could even realize this homeomorphism.

What we do in fact globally have is $H : \mathcal{A} \to [0, 1)$ an extension of $|\phi|$ (I choose to denote this by a different letter to avoid confusion). Recall the definition

$$H(p) = |\phi(f^k(p))|^{1/n^k},$$

for a large enough k so that $f^k(p)$ belongs to the conjugating neighborhood of p (the value will not depend on which k large we choose). Note that $H(p) = 0 \iff f^n(p) = \hat{p}$ for some iterate n, and more generally,

$$H(f(p)) = H(p)^n.$$

In the proof of theorem 9.3, the book claims that if the local inverse ψ_ε to the Böttcher map extends to $\mathbb{D}$, $r = 1$ then $H(p) \to 1$ as $p \to \partial U$. This is due to p escaping all compact sets of U, which can be described in a compact exhaustion by $H^{-1}[0, r]$ for $r < 1$, since we have the conformal isomorphism so that H is a proper map.

Moreover, if the inverse extends only to $\mathbb{D}_r$ for $r < 1$, we necessarily have that $H(p) = r$ for all $p \in \partial U$ by continuity and properness.

Comment on Theorem 9.5:

It is claimed that each of the closures $\overline{\Psi(\mathbb{A}_{1+\varepsilon})}$ contains the Julia set. We prove this for the sake of completeness. Recall that $\Phi : \mathbb{C} \setminus K \to \mathbb{C} \setminus \overline{\mathbb{D}}$ is a conformal isomorphism given by the Böttcher coordinates, with inverse Ψ, and that $J = \partial K = \partial(\mathbb{C} \setminus K)$. If $z \in \partial K$, then any neighborhood of z intersects $\mathbb{C} \setminus K$, which has image by Φ in thin annuli in $\mathbb{C} \setminus \overline{\mathbb{D}}$. This also stems from the properness of the map Φ, and $H(p) \to 1$ as p approaches the boundary.

Later in the proof, we have the isomorphism $\Psi : \mathbb{C} \setminus \overline{\mathbb{D}_r} \to U \subset \mathbb{C} \setminus K$, where $\overline{U} \subset \mathbb{C} \setminus K$ and ∂U contains a critical point c. For $f(c) = v \in U$, $|\Phi(v)| = r^n$, we have the external ray $R' = \Psi(R) \subset U$ landing at v, and the inverse image $f^{-1}(R')$.

If $z \in f^{-1}(R')$, then $f(z) = \Psi(t\Phi(v))$ for some $t \geq 1$, or $\Phi(f(z)) = t\Phi(v)$. If $t > 1$, evidently $z \in U = \Psi(\mathbb{D}_r)$ by seeing that $\Phi(f(z))$ has n-th roots in $\mathbb{C} \setminus \overline{\mathbb{D}_r}$, with conjugation with f. If $t = 1$, so that $\Phi(f(z)) = \Phi(v)$, or more precisely that $f(z) = v$, then any neighborhood N of z has its image $f(N)$ intersecting $R' \setminus \{v\}$ by openess and maximum modulus, so that $z \in \overline{f^{-1}(R' \setminus \{v\})} \subset \overline{U}$.

Moreover, the proof claims that each ray R'_j mapping to R' has to end at a solution of $f(z) = v$. Why? Maybe the reasonings above already give a proof of this claim, but I'm not thoroughly convinced.

Problem (9-a. Harmonic functions)**.** (I will omit the first part of this problem.) Consider a polynomial f of degree $n \geq 2$. Show that the Green's function $G(z) = \log|\Phi(z)|$ is harmonic on $\mathbb{C} \setminus K$, that it tends to zero as z approaches K, and that it satisfies

$$G(z) = \log|z| + O(1) \quad \text{as} \quad z \to \infty.$$

(In other words, $G(z) - \log|z|$ is bounded for large z. A more precise estimate would be $G(z) = \log|z| + \log|a_n|/(n-1) + o(1)$ as $|z| \to \infty$, where a_n is the leading coefficient. Show that the function G is uniquely characterized by these properties. Hence G is completely determined by the compact set $K = K(f)$, although our construction of G depends explicitly on the polynomial f.

Proof. Recall that if $\Phi : U \to \mathbb{C} \setminus \overline{\mathbb{D}_r}$ are the Böttcher coordinates mapping some neighborhood of infinity to another neighborhood of infinity biholomorphically, we may extend $|\Phi(z)|$ to the whole basin of attraction, mapping $\mathcal{A}(\infty)$ to $(1, \infty)$, and therefore $G(z) = \log|\Phi(z)|$ is well defined as a map from $\mathbb{C} \setminus K \to (0, \infty)$. We can equivalently define

$$G(z) = \lim_{k \to \infty} \frac{1}{n^k} \log|f^k(z)|$$

by construction of the Böttcher map.

Recall that $\mathbb{C} \setminus K = \mathcal{A}(\infty)$ is connected, being the unique unbounded component of $\mathbb{C} \setminus J$, and that G satisfies

$$G(f(z)) = n \cdot G(z).$$

Being the logarithm of the absolute value of a non-vanishing holomorphic function, G is harmonic: on any simply connected open set V of the domain, $\Phi : V \to \mathbb{C} \setminus \{0\}$ factors through the exponential map, so that $\Phi(z) = e^{h(z)}$ for some $h(z)$ holomorphic. Therefore $\log|\Phi(z)| = \mathrm{Re}\,(h(z))$, being the real part of a holomorphic function and therefore harmonic.

We show that $G(z)$ approaches 0 as z approaches K. It is useful to consider the neighborhoods of infinity adjoined ∞, where G is continuous by assuming $G(\infty) = \infty$. (One could also use the approach that G is proper.) This implies that the compact sets $\{G(z) \geq s\}$ for $s > 0$ form a compact exhaustion of $\mathcal{A}(\infty)$, as f has no poles in $\mathcal{A}(\infty)$. In order to approach $\partial K = \partial \mathcal{A}(\infty)$, points must leave all compact sets of $\mathcal{A}(\infty)$, so that $G(z) \to 0$.

In order to show that $G(z) = \log|z| + O(1)$ as $z \to \infty$, it is equivalent to show that $|\Phi(z)/z|$ is bounded for all sufficiently large n. But $\Phi(z)/z$, under the change of coordinates $z \mapsto 1/z$, is equivalent to $\phi(z)/z$ as $z \to 0$ in the standard Böttcher coordinates, which approaches $\phi'(0)$. For the polynomial f, we have

$$g(z) = 1/f(1/z) = \frac{1}{a_n z^{-n} + \cdots + a_0} = \frac{z^n}{a_0 z^n + \cdots + a_n},$$

so that in power series form

$$g(z) = z^n(\frac{1}{a_n} - \frac{a_{n-1}}{a_n^2} z + \ldots)$$

By taking c such that $c^{n-1} = 1/a_n$ and conjugating $h(z) = cg(z/c)$ we have

$$h(z) = z^n(1 - \frac{a_{n-1}}{ca_n} z + \ldots)$$

(...)

If H were another harmonic function with these properties, then $G - H$ would be harmonic on $\mathbb{C} \setminus K$, bounded near infinity, and tend to 0 as z approaches K. By the maximum principle, this implies that $G - H$ is constant equal to 0, so that $G = H$.

This implies that G is completely determined by the compact set K, though not every compact set may give rise to a harmonic function in this form. (This may be true, but does not follow from just the above: interesting question. As G arises from K a Julia set, which properties of Julia sets do you need to be true for K to give rise to a harmonic function?) □

OBS.: From the above, in order for $K \subset \mathbb{C}$ compact to have an associated Green's function, satisfying the properties above, we must have that the complement $\mathbb{C} \setminus K$ is connected, by the maximum modulus principle for harmonic functions. For if U were a bounded component of $\mathbb{C} \setminus K$, as $z \to K$ implies $G(z) \to 0$, we would have that $G|_U \equiv 0$, a contradiction with $G > 0$.

Moreover, K must not have isolated points, otherwise by the removing the singularity due to continuity we would have $G(p) = 0$ at that point, another contradiction.

Problem (9-c. Cellular sets and Riemann-Hurwitz)**.** Let f be a polynomial of degree $n \geq 2$. For each number $g > 0$ let V_g be the bounded open set consisting of all complex numbers z with $G(z) < g$. Using the maximum modulus principle, show that each connected component of V_g is simply connected. Hence the Euler characteristic $\chi(V_g)$ can be identified with withe number of connected components of V_g. Show similarly that each component of V_g intersects the filled Julia set.

The Riemann-Hurwitz formula applied to the map $f : V_g \to V_{ng}$ asserts that $n\chi(V_{ng}) - \chi(V_g)$ is equal to the number of critical points of f in V_g, counted with multiplicity, conclude that V_g is connected if and only if it cointains all of the $n-1$ critical points of f.

A compact subset of Euclidean n-space is said to be *cellular* if it is a nested intersection of closed topological n-cells, each containing the next in its interior. Show that the filled Julia set $K = \bigcap V_g$ is cellular (and hence connected) if and only it contains all of the $n-1$ finite critical points of f. In fact, if one of these critical points lies outside of K, and hence outside of some V_g, show that V_g and hence K is not connected.

Proof. First we see that the set $\{G(z) > g\}$ is connected; naturally it contains a neighborhood of infinity, and if it had a bounded component W, by the maximum principle, $G|_{\overline{W}}$ attains a maximum at $\partial W \subseteq \{G(z) \leq g\}$, a contradiction. This implies that $\{G(z) \geq g\} = \overline{\{G(z) > g\}} = \mathbb{C} \setminus V_g$ is also connected.

Let U be a connected component of V_g, hence open and bounded. We show that $\mathbb{C} \setminus U$ is connected, so that U is simply connected. Naturally $\mathbb{C} \setminus U$ has a unique unbounded component, so we must show $\mathbb{C} \setminus U$ has no bounded component. Suppose A is a bounded component of $\mathbb{C} \setminus U$ (hence compact).

Since $\mathbb{C} \setminus V_g \subseteq \mathbb{C} \setminus U$, and as $\mathbb{C} \setminus V_g$ is connected, we actually have that $\mathbb{C} \setminus V_g$ is contained in the unique unbounded component of $\mathbb{C} \setminus U$, so that $(\mathbb{C} \setminus V_g) \cap A = \emptyset$, and therefore $A \subset V_g$. However, $\partial A \subseteq \partial U \subseteq \partial V_g$, a contradiction with A compactly contained in V_g. Hence $\mathbb{C} \setminus U$ is connected, being the unbounded component, and U is simply connected.

Suppose now that U a component of V_g does not intersect the filled Julia set K. This implies that G is harmonic on U, having maximum and minimum on $\partial U \subseteq \partial V_g = \{G(z) = g\}$, so that G is constant throughout U, a contradiction.

Suppose now that V_g is connected, so that $\chi(V_g) = 1$. As

$$\#\mathrm{crit}(f) = n\chi(V_{ng}) - \chi(V_g)$$

is the number of critical points of $f|_{V_g} : V_g \to V_{ng}$ counted with multiplicity, we have $\#\mathrm{crit}(f) \geq n - 1$. But f, considered as a polynomial on $\mathbb{C}$, has at most $n-1$ critical points. Hence $\#\text{crit}(f) = n-1$, and V_g contains all $n-1$ (finite) critical points of f. Conversely, if V_g contains all the finite critical points, so that

$$n - 1 = n\chi(V_{gn}) - \chi(V_g),$$

and $\chi(V_g) > 1$, then $\chi(V_{ng}) > 1$. But all of the $n-1$ critical points of f are still in V_{ng}, so we repeat this argument for $f : V_{ng} \to V_{n^2 g}$ to conclude that $\chi(V_{n^2 g}) > 1$. But since for all sufficiently big g we have that V_g is simply connected, so $\chi(V_g) = 1$, this leads to a contradiction. Hence $\chi(V_g) = 1$.

If K contains all of the $n-1$ finite critical points of f, then from the above, V_g is connected for all $g > 0$. Naturally for $g < g'$ we have $V_{g'}$ compactly contained in V_g, and we have $K = G^{-1}(0) = \bigcap V_g = \bigcap \overline{V_g}$, the nested intersection of closed topological 2-cells.

Conversely, if at least one critical point lies outside of K, then for some V_g we have $\chi(V_g) > 1$ and for all $g' \leq g$, having more than one component, so the nested intersection $\bigcap \overline{V_g}$ cannot be connected. More explicitly, let U and V be two disjoint components of V_g. We can show that $U \cap \bigcap_{g'<g} \overline{V_{g'}}$ is nonempty, as well as for V; otherwise $U \cap K = \emptyset$, a contradiction. □

Problem (9-d. Quadratic polynomials)**.** Nw let $f(z) = z^2 + c$, and suppose that the critical orbit escapes to infinity. Let $V = V_{G(c)}$ be the open set consisting of all $z \in \mathbb{C}$ with $|\Phi(z)| < |\Phi(c)|$. Show that V is conformally isomorphic to $\mathbb{D}$, and that $f^{-1}(V)$ has two connected components. Conclude that $f^{-1}|_V$ has two holomorphic branches g_0 and g_1 mapping V into disjoint open subsets, each having compact closure in V. Show that g_j strictly contracts the Poincaré metric of V. Proceeding as in problem 4-e, show that J is a Cantor set, canonnically homeomorphic to the space of all infinite sequences $(j_0, j_1, j_2, \ldots)$ of zeros and ones.

Proof. Note that 0 is the unique (finite) critical point of f, and since $G(0) > 0$, $G(c) = G(f(0)) = 2G(0) > G(0)$, so that $V = V_{G(c)}$ contains 0. From the previous exercise, $\mathbb{C} \setminus V$ is connected and V is simply connected. By the Riemann mapping theorem V is conformally isomorphic to $\mathbb{D}$. From the Riemann-Hurwitz formula, and since $0 \notin V_{G(0)}$, we have for $f : V_{G(0)} \to V_{G(c)}$:

$$0 = 2\chi(V_{G(c)}) - \chi(V_{G(0)}) \implies \chi(V_{G(0)}) = 2.$$

Note also that for all $g > G(0)$, we still have V_g connected, and so $\overline{V_g}$ connected. As $\overline{V_{G(0)}} = \bigcap_{g>G(0)} \overline{V_g}$, we in fact have that $\overline{V_{G(0)}}$ is connected, despite $V_{G(0)}$ having two components.

(If we repeat the arguments in the proof of 9.5 in the book, we have that two external rays will separate the two components of $V_{G(0)}$, meeting at exactly the critical point 0. Maybe one can work a little more to show that $\partial V_{G(0)}$ is a figure 8 with self-intersection at 0.)

We see that the two components A and B of $V_{G(0)}$ are interchanged by $z \mapsto -z$. Evidently this involution preserves $V_{G(0)}$ since $f(-z) = f(z)$. By connectedness, if it preserved one component, we could find a path from z to $-z$ in A , and then taking the image of this path by $z \mapsto -z$, this creates a loop around 0 in A. But A is simply connected and does not contain 0, a contradiction. This shows not only that each V_g is centrally symmetric, but that the two components of $V_{G(0)}$ are interchanged by $z \mapsto -z$.

As

$$G(f(z)) < G(c) \iff G(z) < G(0),$$

we can see that $f^{-1}(V) = V_{G(0)}$ has two components. As V does not contain the critical value c and is simply connected, we can extend a the local inverses of $f|_{V_{G(0)}} : V_{G(0)} \to V$ to all of V, each mapping to one of the two components of $V_{G(0)}$ necessarily, since both intersect K and if g_0 is an inverse, then $g_1 = -g_0$ mapping to the other component is another inverse.

Each g_j must strictly contract the hyperbolic metric since the components of $V_{G(0)}$ are compactly contained in V, and the rest follows by the choices of inverses we take by iteration. □

10 Parabolic Fixed Points: the Leau-Fatou Flower

Comment on Definition 10.2 and proof of 10.1:

We can see from the proof of lemma 10.1 that, for every $\varepsilon > 0$, the basin of attraction $\mathcal{A}_j = \mathcal{A}(\hat{z}, v_j)$ will necessarily contain a sector around $\hat{z}$ of the form $\delta v_j e^{i\theta}$, where $|\theta| < \pi/n - \varepsilon$ and $\delta = \delta(\varepsilon) > 0$ depends on ε. Any point converging non-trivially to $\hat{z}$ not only must be contained in one of these basins (by definition and by lemma 10.1), but must also eventually be fully contained in one of these sectors.

The connected component $\mathcal{A}_j^0$ of $\mathcal{A}_j$ which contains one (and hence all) of these sectors must then be invariant under f, and be the unique component of $\mathcal{A}_j$ invariant under f, as all others must eventually map into it. Given the proof of 10.1, we see that $A_j \subseteq F(f)$ because on these small enough sectors, compact sets converge uniformly to the fixed point $\hat{z}$.

Remark on the proof of 10.4: For a local expression $f(z) = \lambda z + O(z^2)$, where $\lambda = e^{2\pi i p/q}$ is a primitive q-th root of unity and fixed point $\hat{z}$, we see that $\hat{z}$ must be a simple fixed point of f^k for all $k < q$ because

$$f^k(z) = \lambda^k z + O(z^2),$$

and $\lambda^k \neq 1$. When we perturb $f = f_0$ generically to some f_t, the $(n+1)$-fold fixed point of f_0^q breaks apart into $n+1$ simple fixed points for f_t^q. We see that $\hat{z}$ must still be perturbed to a simple fixed point $\hat{z}_t$ of f_t; It will be the unique fixed point of f_t in a small neighborhood. Moreover, the simple fixed point $\hat{z}$ of f_0^k will be preserved to a simple fixed point of f_t^k, which must in fact be the point $\hat{z}_t$.

All the other n simple fixed points of f_t are periodic of period q, and since they cannot be fixed points for f_t^k for $k < q$, the period is exactly q. Hence they are arranged into n/q orbits of period exactly q.

Question: How many of these orbits can be attracting/repelling?

Comment on Petals:

Note that from the definition of an attracting petal $\mathcal{P}$, we have that $\mathcal{P} \subset \mathcal{A}_j$ for the attracting direction v_j, but $\mathcal{P}$ can possibly not be the full immediate basin $\mathcal{A}_j^0$, since we ask that the converge be uniform in $\mathcal{P}$, not just locally uniform, and for points nearby $\hat{z}$ in $\mathcal{A}_j^0$ but close to the repelling directions, the convergence is not uniform (consider a neighborhood of infinity for $z \mapsto z+1$).

Must $\mathcal{A}_j^0$ be simply connected? This is an exercise later on.

Does this stem from a more general fact that for a polynomial, the Fatou components are already simply connected?

Proposition 10.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial and U a bounded Fatou component of f. Then U is simply connected.*

The proof should follow from the maximum principle: a point in the Julia set "inside" of U could not be mapped to the boundary "outside" of U, for it would contradict the maximum principle in some form. Maybe a more precise and more general statement is that if $\widetilde{U}$ is U along with all of its holes, then f maps $\widetilde{U}$ to $\widetilde{U}$.

(I saw a proof: apply the maximum principle for $|f^n|$.)

Problem (10-a. Repelling petals and the Julia set)**.** If f is a non-linear rational function, then every repelling petal must intersect the Julia set of f.

Proof. Suppose that some repelling petal did not intersect the Julia set of f, so that it is contained in the Fatou set. But since it intersects the immediate attracting basins to either side of it, it must coincide with them as components of the Fatou set. But then the attracting directions for these attracting petals must be the same, so that the multiplicity is $n = 1$. But this shows that $\hat{z}$ is an isolated point of the Julia set, a contradiction with f being non-linear. □

Problem (10-b. No small cycles)**.** Show that no orbit $z_0 \mapsto z_1 \mapsto \cdots$ under f can be contained in the union $\mathcal{P}'_1 \cup \cdots \cup \mathcal{P}'_n$ of repelling petals. Conclude that the only periodic orbit which is completely contained in the neighborhood

$$N_0 = \{\hat{z}\} \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_n \cup \mathcal{P}'_1 \cup \cdots \cup \mathcal{P}'_n$$

is the fixed point $\hat{z}$ itself. On the other hand, show that any non-linear rational function has orbits which return to every repelling petal infinitely often.

Proof. Suppose that some periodic orbit $z_0, \ldots, z_{m-1}, z_m = z_0$ is contained in the union of repelling petals. By considering $g = f^m$, if $f(z) = z + az^{n+1} + \ldots$, we have that

$$g(z) = z + maz^{n+1} + \ldots,$$

so that $\hat{z}$ is still a parabolic fixed point with multiplier $\lambda = 1$ and multiplicity $n+1$. The attracting and repelling petals for $f^m = g$ are the same ones for f. In this case, we may assume that z_0 is a fixed point contained in one of the repelling petals. But by the Abel functional equation on the repelling petals, they cannot admit fixed points.

□

Problem (10-e. Immediate parabolic basins)**.** By an argument similar to that of 8.7, show that the complement of an immediate parabolic basin is either connected or else has uncountably many connected components.

Proof. We repeat the ideas of 8.7. Let $\mathcal{P}$ be an attracting petal within an immediate parabolic basin $\mathcal{A}^0$ such that $f(\overline{\mathcal{P}}) \subseteq \mathcal{P} \cup \{\hat{z}\}$, and such that $\partial\mathcal{P} \setminus \{\hat{z}\}$ does not intersect the postcritical set of f. Let $\mathcal{P}_k$ be the connected component of $f^{-k}(\mathcal{P})$ which contais $\mathcal{P}$, so that $\mathcal{P}_k \subset \mathcal{P}_{k+1}$ and $\mathcal{A}^0 = \bigcup_{k\geq 0} \mathcal{P}_k$.

Note that $\partial\mathcal{P}$ can be taken to be a simple closed curve, such that $\partial\mathcal{P} \cap (\hat{\mathbb{C}} \setminus \mathcal{A}^0) = \{\hat{z}\}$. Moreover, note that $\hat{z}$ has no preimages in $\mathcal{A}^0$ by virtue of non-trivial convergence. Each $\mathcal{P}_k$ is then bounded by some finite number of simple closed curves, only one of which contains $\hat{z}$. We then reduce to the case of whether each $\mathcal{P}_k$ is bounded by one simple closed curve, so that $\hat{\mathbb{C}} \setminus \mathcal{A}^0$ is connected as a nested intersection of connected sets, or if some $\mathcal{P}_k$ has two or more simple closed curves bounding it.

From 8.7, if $\Gamma_1, \ldots, \Gamma_m$ are the simple closed curves bounding $\mathcal{P}_k$, and if C is a connected component of $\mathcal{P}_{2k} \backslash \overline{\mathcal{P}_k}$, then $C \to \mathcal{P}_k \backslash \overline{\mathcal{P}}$ is a branched covering. $\mathcal{P}_k \setminus \overline{\mathcal{P}}$ has $m+1$ boundary curves, so C must have $m+1$ boundary curves (as the boundaries of each $\mathcal{P}_k$ avoid postcritical points). Therefore each of the m components of $\hat{\mathbb{C}} \setminus \mathcal{P}_k$ contain at least m connected components of $\hat{\mathbb{C}} \setminus \mathcal{P}_{2k}$.(?)

□

Problem (10-f. Examples with a parabolic point at infinity)**.** Prove:

(1) For $f(z) = z - 1/z$ show that there is a parabolic point at infinity with two attracting directions. Since the upper and lower half-planes map into themselves, conclude that $J = \mathbb{R} \cup \{\infty\}$.

(2) For $f(z) = z - 1/z + 1$ show that J is a Cantor set contained in $\mathbb{R} \cup \{\infty\}$.

(3) For $f(z) = z + 1/z - 2$ show that J is the interval $[0, +\infty]$.

(4) For $f(z) = z + 1/(1+z^2)$ show that there are three attracting directions at infinity. Show that one of the three immediate parabolic basins contains all of $\mathbb{R}$, and hence *nearly* disconnects the Riemann sphere.

Proof. (1) Locally around infinity, we have the map

$$g(w) = 1/f(1/w) = \frac{1}{1/w - w} = \frac{w}{1 - w^2} = w + w^3 + w^5 + \dots$$

so that the multiplicity of $w = 0$ as a parabolic fixed point is $2 + 1$, having two attracting directions. The upper and lower half planes are preserved under iteration of f, so that by normality they must be contained in the Fatou set, and $J \subseteq \mathbb{R} \cup \{\infty\}$. The upper and lower half planes must intersect the two immediate basins of attractions of ∞, and hence must coincide with them. As they must be disjoint, we must have that $J = \mathbb{R} \cup \{\infty\}$.

(2) By the change of coordinates $w = 1/z$, ∞ is a parabolic fixed point with multiplier $n + 1 = 2$, because

$$\frac{1}{\frac{1}{w} - w + 1} = \frac{w}{1 + w - w^2} = w + Cw^2 + \cdots$$

where $C \neq 0$, so that there is a unique immediate attracting basin at infinity. By the previous arguments, the upper and lower half planes are invariant under f and must be contained in the Fatou set, so $J \subseteq \mathbb{R} \cup \{\infty\}$.

The Julia set must have no isolated points and perfect, so we need only prove that $J(f)$ contains no intervals. Moreover, we see that $f(-1/z) = f(z)$, so that $z \in J(f) \iff -1/z \in J(f)$.

Note that points in $(1, +\infty)$ converge non-trivially to ∞, as f is increasing and has no fixed points on this interval. This also implies that $(-1, 0)$ is contained in the Fatou set. If J contained an interval, it would have to contain an interval in $[0,1]$. The derivative of f everywhere is greater than 1, so the interval gets expanded under f. If we take an interval of biggest length of $\operatorname{int} J \cap [0,1]$, where eventually it must be mapped back into $[0,1]$, we obtain a contradiction. Hence $J \cap \mathbb{R}$ must be a Cantor set.

(3) In this case, by the change of coordinates,

$$\begin{aligned} \frac{1}{\frac{1}{w}+w-2} = \frac{w}{1-2w+w^2} = \frac{w}{(1-w)^2} \\ = w(1-w+w^2+\cdots)(1-w+w^2+\cdots) \\ = w - 2w^2 + 3w^3 + \ldots, \end{aligned}$$

so that the multiplicity of ∞ is $2 = 1+1$, having only one attracting direction. Note that $f(z) = f(1/z)$, so that $z \in J(f) \iff 1/z \in J(f)$.

For $z \leq 0$, we see that $f(z) \leq z - 2$, so that $f^n(z) \leq z - 2n$ and z converges to ∞ non-trivially, so that $(-\infty, 0) \subset F(f)$. In fact, for $\operatorname{Re} z < 0$, we see that

$$\operatorname{Re} f(z) = \operatorname{Re} z + \operatorname{Re}(1/z) - 2 < \operatorname{Re} z - 2,$$

so that we also have $z \to \infty$ non-trivially and the left half-plane contained in the Fatou set.

Note that if $w \geq 0$, then the two preimages of w are non-negative real numbers. This implies that $\mathbb{R}_+ \cup \{\infty\}$ is backwards invariant under f (and in fact fully invariant); since $f(0) = \infty$ and $0 \in J(f)$, we have that $J(f) \subseteq [0, +\infty]$. By connectedness of $\hat{\mathbb{C}} \setminus [0, +\infty] \subseteq F(f)$, this must be contained in a single component of $F(f)$, being the immediate attracting basin for ∞. If for some $z \in (0, +\infty)$ we had $z \in F(f)$, it would have to converge non-trivially to infinity. But this evidently cannot happen due to the dynamics of f in $\mathbb{R}^+$, where every orbit under iteration eventually falls within the interval $[0,2]$, mapping to infinity, staying within the interval or returning to it infinitely often.

(4) Just calculate the coordinate change $w = 1/z$, check that the multiplicity is $4 = 3+1$, and its easy to see that points in $\mathbb{R}$ converge to infinity non-trivially within an immediate attracting basin.

□

11 Cremer Points and Siegel Disks

Problem (Generic Angles). Given a completely arbitrary sequence of positive real numbers $\epsilon_1, \epsilon_2, \ldots \to 0$ decreasing monotonically, let $S(q_0)$ be the set of all real numbers ξ such that

$$|\xi - p/q| < \epsilon_q$$

for some fraction p/q in lowest terms with $q > q_0$. Show that $S(q_0)$ is a dense open subset of $\mathbb{R}$, and conclude that the intersection $\bigcap_{q_0} S(q_0)$, consisting of all ξ for which this condition is satisfied for inifinitely many p/q, is a countable intersection of dense open sets. As an example, taking $\epsilon_q = 2^{-q!}$ conclude that a generic real number belongs to the set S and hence satisfies Cremer's condition that $\liminf |\lambda^q - 1|^{1/d^q} = 0$ for every degree d.

Proof. Evidently $S(q_0)$ is the union of all open intervals $(p/q - \varepsilon_q, p/q + \varepsilon_q)$ over $p \in \mathbb{Z}$ and $q > q_0$ such that $\gcd(p, q) = 1$, so that it is open. Let $I \subset \mathbb{R}$ be some non-empty open interval. Consider $q > q_0$ for which $\epsilon_q < \frac{1}{2}|I|$. As $\mathbb{Q}$ is dense in $\mathbb{R}$, there will exist some $p'/q' \in \mathbb{Q}$ with $q' > q$ and $p'/q' \in \frac{1}{2}I$, where here $\frac{1}{2}I$ represents the open interval with the same center as I but half the radius. But this implies that $(p'/q' - \epsilon_{q'}, p'/q' + \epsilon_{q'}) \subset I$, Showing that $S(q_0) \cap I \neq \emptyset$. This concludes that $S(q_0)$ is dense open. By the Baire category theorem, S is dense and uncountable.

If we consider $\epsilon_q = 2^{-q!}$, then S consists of all real numbers ξ for which there are infinitely many fractions p/q in lowest terms such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{2^{q!}}.$$

Since then $|\lambda^q - 1| \sim |q\xi - p| < q/2^{q!}$ up to a fixed multiplicative constant, we have that

$$\log |\lambda^q - 1|^{1/d^q} < \frac{1}{d^q} \log \frac{q}{2^{q!}} = \frac{\log q}{d^q} - \frac{\log 2^{q!}}{d^q} = \frac{\log q}{d^q} - \log 2 \frac{q!}{d^q},$$

where $(\log q)/d^q \to 0$ as $q \to \infty$. However, $q!/d^q \to \infty$ as $q \to \infty$, so the above expression has limit $-\infty$, and

$$\lim |\lambda^q - 1|^{1/d^q} = 0,$$

for all $d \in \mathbb{N}$. □

Problem (Cremer's 1938 theorem). If $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots$, where λ is not zero and not a root of unity, show that there is one and only one formal power series of the form $h(z) = z + h_2 z^2 + h_3 z^3 + \cdots$ which formally satisfies the condition that $h(\lambda z) = f(h(z))$. In fact

$$h_n = \frac{a_n + X_n}{\lambda^n - \lambda}$$

for $n \geq 2$, where $X_n = X(a_2, \ldots, a_{n-1}, h_2, \ldots, h_{n-1})$ is a certain polynomial expression whose value can be computed inductively. Now suppose that we choose the a_n inductively, always equal to zero or one, so that $|a_n + X_n| \geq 1/2$. If

$$\liminf_{q\to\infty} |\lambda^q - 1|^{1/q} = 0,$$

show that the uniquely defined power series $h(z)$ has radius of convergence zero. Conclude that $f(z)$ is a holomorphic germ which is not locally linearizable. Choosing the a_n more carefuly, show that we can even choose $f(z)$ to be an entire function.

Problem. First we see that if $\liminf |\lambda^q - 1|^{1/q} = 0$, then we also have $\liminf |\lambda^q - \lambda|^{1/q} = 0$ (why?). We have

$$h(\lambda z) = \lambda z + \lambda^2 h_2 z^2 + \lambda^3 h_3 z^3 + \cdots ,$$

and

$$\begin{aligned} f(h(z)) &= \lambda \left(\sum_{i_1}^{\infty} h_{i_1} z^{i_1} \right) + a_2 \left(\sum_{i_2}^{\infty} h_{i_2} z^{i_2} \right)^2 + a_3 \left(\sum_{i_3}^{\infty} h_{i_3} z^{i_3} \right)^3 + \cdots \\ &= \lambda h_1 z + (\lambda h_2 + a_2 h_1^2) z^2 + (\lambda h_3 + 2a_2 h_1 h_2 + a_3 h_1^3) z^3 + \cdots \end{aligned}$$

where the general term $H_n z^n$ is given by

$$H_n = \sum \lambda h_{i_1} + a_2 h_{i_{2,1}} h_{i_{2,2}} + a_3 h_{i_{3,1}} h_{i_{3,2}} h_{i_{3,3}} + \ldots + a_n h_{i_{n,1}} \ldots h_{i_{n,n}})$$

where the sum is over all possible choices of indices $i_1, i_{2,j}, \ldots, i_{n,j}$ such that $\sum_{l=1}^{j} i_{j,l} = n$. Here we have $h_1 = 1$. We also have that the first term of H_n is λh_n and the last is a_n, so that by equating $H_n = \lambda^n h_n$, we get the the above uniquely defined expression for h_n, $n \geq 2$, since λ is not a root of unity.

If we have choose the a_n inductively so that $|a_n + X_n| \geq 1/2$ for all $n \geq 2$, then

$$|h_n| \geq \frac{1}{2|\lambda^n - \lambda|} \implies |\lambda^n - \lambda| \geq \frac{1}{2|h_n|}.$$

As $\liminf |\lambda^q - \lambda|^{1/q} = 0$, we have that $\liminf (1/|h_q|)^{1/q} = 0$, and

$$\limsup_{q\to\infty} |h_q|^{1/q} = \infty.$$

This implies that $f(z)$ is a holomorphic germ which is not locally linearizable, because if it were, the linearizing map would have to be h as given above by the uniqueness of its coefficients, but h has radius of convergence zero.

If we want to inductively choose the coefficients a_n of f so that the radius of convergence of f is infinite, we must have $\limsup_{n\to\infty} |a_n|^{1/n} = 0$ simultaneously with $\limsup |h_n|^{1/n} = \infty$. (How to proceed?)

Problem (11-d. Small Cycles)**.** Suppose that

$$\limsup_{q\to\infty} \frac{\log\log(1/|\lambda^q - 1|)}{q} > \log d > 0.$$

Modify the proof of 11.2 to show that: Any fixed point of multiplier λ for a rational function f of degree d has the small cycles property. First choose $\varepsilon > 0$ so that

$$\log\log(1/|\lambda^q - 1|) > (\varepsilon + \log d)q,$$

or equivalently

$$|\lambda^q - 1|^{1/d^q} < \exp(-e^{\varepsilon q})$$

for infinitely many q. The proof of 11.2 then constructs points z_q of period q with $|z_q| < \exp(-e^{\varepsilon q})$. Now use Taylor's theorem to find $\delta > 0$ so that $|f(z)| < e^{\varepsilon}|z|$ for $|z| < \delta$, hence $|f^q(z)| < \delta$ for $|z| < e^{-q\varepsilon}\delta$. Finally, note that $\exp(-e^{-\varepsilon q}) < e^{-q\varepsilon}\delta$ for large q, and conclude that f has small cycles.

Proof. Theorem 11.2 is recalled:

Theorem 11.1 (Cremer Non-linearization)**.** *Given $\lambda \in S^1$ and given $d \geq 2$, if the d^q-th root of $1/|\lambda^q - 1|$ is unbounded as $q \to \infty$, then no fixed point of multiplier λ for a rational function f of degree d can be locally linearizable.*

In the proof, we may assume the fixed point is at the origin. The proof is generally algebraic, where the fixed points for f^q satisfy certain algebraic equations and at least one must have small absolute value. As mentioned above, we construct a periodic point z_q of period q with $|z_q| < \exp(-e^{\varepsilon q})$. By the definition of f' at $z = 0$ as a limit, since $|f'(0)| = 1$, there is some $\delta > 0$ so that if $0 < |z| < \delta$, then $|f(z)/z| < e^{\varepsilon}$, and $|f(z)| < e^{\varepsilon}|z|$.

For $|z| < e^{-q\varepsilon}\delta$, applying the above result k times for $1 \leq k \leq q$, we have $|f^k(z)| < e^{(k-q)\varepsilon}\delta \leq \delta$.

As $\dfrac{\exp(-x)}{\delta x} \to 0$ as $x \to 0$, for sufficiently big q we have $\exp(-e^{-q\varepsilon}) < e^{-q\varepsilon}\delta$, so that the periodic point z_q constructed is such that $|f^k(z_q)| < \delta$, having its entire orbit contained in this $\delta > 0$ neighborhood. By taking $\varepsilon > 0$ arbitrarily small and $\delta \to 0$ correspondingly, we find small cycles close to the origin in every neighborhood. $\square$

12 The Holomorphic Fixed Point Formula for Rational Maps

Comment on multiplicity:

The multiplicity of a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ at a point $z_0 \in \mathbb{C}$ is defined to be the unique integer $m \geq 1$ such that

$$f(z) - z = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots$$

and $a_m \neq 0$, by taking the power series expansion around z_0. Indeed, z_0 is a fixed point if and only if $m \geq 1$, and $m \geq 2$ if and only if the multiplier λ of f at the fixed point z_0 is $\lambda = 1$. For completeness, also recall that the multiplier λ at a fixed point z_0 is given by

$$f(z) = z_0 + \lambda(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

Indeed, by conjugating f by translation $z = w + z_0$, which amounts to moving z_0 to the origin, we have

$$f(w) = \lambda w + a_2 w^2 + \cdots$$

around a neighborhood of 0. Both the notion of the multiplier and the multiplicity may be extended to $\infty \in \hat{\mathbb{C}}$ by using the coordinate chart $z \mapsto 1/z$ around ∞.

In fact, these notions may be generalized to any Riemann surface S and a fixed point $\hat{z}$ on S for $f : S \to S$. One way of seeing that the multiplier is well defined at a fixed point is that the differential

$$df_{\hat{z}} : T_{\hat{z}}S \to T_{\hat{z}}S$$

is a $\mathbb{C}$-linear transformation on a 1-dimensional $\mathbb{C}$-vector space, hence it must be multiplication by a unique complex number.

Given a coordinate chart $\varphi : U \subseteq S \to \mathbb{C}$ such that $\varphi(\hat{z}) = 0$, we find

$$\varphi \circ f \circ \varphi^{-1}(z) = g(z) = \lambda z + a_m z^m + a_{m+1} z^{m+1} + \cdots ,$$

where a_m is the first non-zero term after λz. Consider ψ another coordinate chart such that $\psi(\hat{z}) = 0$ and

$$\psi \circ f \circ \psi^{-1}(z) = h(z) = \eta z + b_l z^l + b_{l+1} z^{l+1} + \cdots$$

We have that $h = \phi \circ g \circ \phi^{-1}$, where $\phi = \varphi \circ \psi^{-1}$ is a conformal map on a neighborhood of 0 fixing 0. We just need to show that the multiplier and the multiplicity are then invariant under this conformal change of coordinates.

First,

$$\eta = h'(0) = (\phi \circ g \circ \phi^{-1})'(0) = \phi'(0) g'(0) \phi'(0)^{-1} = g'(0) = \lambda.$$

In fact, since we may pullback the differential $df_{\hat{z}}$ to a linear map $dg_0 : T_0\mathbb{C} \to T_0\mathbb{C}$, where we identify $T_0\mathbb{C} \cong \mathbb{C}$, it will indeed act as multiplication by λ.

If we now consider

$$\phi(z) = c_1 z + c_2 z^2 + \dots,$$

and $h(\phi(z)) = \phi(g(z))$, by comparing coefficients, we have

$$\begin{aligned} &h(c_1 z + c_2 z^2 + \cdots) \\ &= \lambda(c_1 z + c_2 z^2 + \cdots) + b_l(c_1 z + c_2 z^2 + \cdots)^l + \cdots \\ &= \lambda c_1 z + \lambda c_2 z^2 + \cdots + \lambda c_{l-1} z^{l-1} + (\lambda c_l + b_l c_1) z^l + \cdots \end{aligned}$$

and

$$\begin{aligned} &\phi(\lambda z + a_m z^m + \cdots) \\ &= c_1(\lambda z + a_m z^m + \cdots) + c_2(\lambda z + a_m z^m + \cdots)^2 + \cdots \\ &= \lambda c_1 z + \lambda c_2 z + \cdots + \lambda c_{m-1} z^{m-1} + (\lambda c_m + c_1 a_m) z^m + \dots \end{aligned}$$

so that the indices l and m and the values b_l and a_m must in fact agree, given that $c_1 \neq 0$.

We also recall that if m is the multiplicity of a parabolic fixed point $\hat{z}$ with multiplier 1, then there are $m-1$ attracting directions and $m-1$ repelling directions at $\hat{z}$.

Later on in the chapter, it is also proved that the *residue fixed point index*

$$\iota(f, z_0) = \frac{1}{2\pi i} \int \frac{dz}{z - f(z)}$$

is well defined on arbitrary Riemann surfaces.

On an open subset $U \subseteq \mathbb{C}$ and $f : U \to \mathbb{C}$ holomorphic such that $f(z_0) = z_0$, if $m \geq 2$, so that $\lambda = 1$, it is also possible to express the multiplicity of f as

$$m = \operatorname{ord}_{z_0}(z - f(z)) = \operatorname{Res}_p((1 - f'(z))/(z - f(z)) = \frac{1}{2\pi i} \int_\gamma \frac{1 - f'(z)}{z - f(z)} dz,$$

where γ is any sufficiently small loop encircling z_0.

Problem (12-a)**.** If $f(z) = z + \alpha z^2 + \beta z^3 + (\text{higher terms})$, with $\alpha \neq 0$, show that the residue index is given by $\iota(f, 0) = \beta/\alpha^2$. As an example, consider the one parameter family of cubic maps

$$f_\alpha(z) = z^3 + \alpha z^2 + z$$

with a double fixed point at the origin. Using 12.4 or by direct calculation, show that the remaining finite fixed point $z = -\alpha$ has a multiplier $\lambda = 1+\alpha^2$, and hence is attracting if and only if α lies within a figure eight shaped region bounded by a lemniscate. This lemniscate is clearly visible as the boundary of the main upper and lower regions in Figure 26, which shows the α-parameter plane. Now, for $\alpha \neq 0$, suppose that we perturb f_α to a map $z \mapsto z^3 + \alpha z^2 + (1-\varepsilon)z$, ao that the double fixed point at the origin splits up into two distinct nearby fixed points. First suppose that α^2 liesinside the disk of radius $1/2$ centered at $1/2$, or equivalently that α lies within a corresponding region bounded by a lemniscate shaped like the symbol ∞. (This has been drawn in as the dotted line in Figure 26.) Show that we can choose a small $\varepsilon \in \mathbb{C}$ so that both of the fixed points near zero are attracting. On the other hand, if α lies strictly outside this region, show for any such perturbation that at least one of the fixed points near zero must be repelling.

Proof. We have

$$\begin{aligned} f(z) &= z + \alpha z^2 + \beta z^3 + \cdots \\ \Longrightarrow\ z - f(z) &= -\alpha z^2 - \beta z^3 - \cdots \\ \Longrightarrow\ \frac{z - f(z)}{z^2} &= -\alpha - \beta z - \cdots \\ \Longrightarrow\ \frac{z^2}{z - f(z)} &= \frac{1}{-\alpha} + \frac{\beta}{\alpha^2} z + \cdots\ , \end{aligned}$$

since we have to calculate the derivative of the inverse. Hence

$$\frac{1}{z - f(z)} = -\alpha \frac{1}{z^2} + \frac{\beta}{\alpha^2}\frac{1}{z} + \cdots\ ,$$

so that the residue is indeed β/α^2. (In order to generalize this index to possibly higher order non-vanishing coefficients of f, we would need to calculate higher derivatives of the inverse of $(z - f(z)/z^k$.)

For the cubic family f_α, the fixed point at $z = 0$ has multiplier 1 and multiplicity 2, being parabolic independently of α, where we may calculate the residue fixed point index $\iota(f_\alpha, 0) = 1/\alpha^2$. As the sum of indices of the finite fixed points of f_α must be 0, we have

$$\iota(f_\alpha, -\alpha) = -\frac{1}{\alpha^2} = \frac{1}{1-\lambda},$$

where λ is its multiplier, since the fixed point must be simple. Hence $\lambda = 1 + \alpha^2$. This point is attracting if and only if $|1+\alpha^2| = |\alpha^2 - (-1)| < 1$, so that α^2 lies in a unit disk centered at -1. The resulting figure for α is the lemniscate; it is the preimage of this disk under the squaring map, so that they are simply connected, one is bounded by the angles $\pi/4 < \theta < 3\pi/4$, and the other by $5\pi/4 < \theta < 7\pi/4$.

We perturb $f_\alpha(z)$ to $f_{\alpha,\varepsilon}(z) = z^3 + \alpha z^2 + (1-\varepsilon)z$. The equation for the fixed points of the perturbed map is

$$z^3 + \alpha z^2 - \varepsilon z,$$

and one of the points is still 0. Its multiplier is $1-\varepsilon$, so that $\iota(f_{\alpha,\varepsilon}, 0) = 1/\varepsilon$. For the other fixed point of f_α, which persists when perturbed, its index varies holomorphically with ε, being $-1/\alpha^2$ for $\varepsilon = 0$.

For α^2 in the disk of radius $1/2$ centered at $1/2$, $1/\alpha^2$ lies in the half-plane $\operatorname{Re} z > 1$, hence $-1/\alpha^2$ lies in the halfplane $\operatorname{Re} z < -1$. For all sufficiently small ε, we may still guarantee that the index $-1/\alpha^2$ has real part $\operatorname{Re}(-1/\alpha^2) \leq -c < -1$, so we only need to take ε so that $1/\varepsilon$ has real part not much bigger than $1/2$.

Explicitly, if $z_0 = 0$ and z_1 are the fixed points near 0 with residues $\iota_0 = 1/\varepsilon$ and ι_1, so that $\operatorname{Re}\iota_0 + \operatorname{Re}\iota_1 \geq c > 1$, We want

$$1/2 < \operatorname{Re}\iota_0 < c/2,$$

so that $\operatorname{Re}\iota_1 > 1/2$. But this is always possible, by taking $1/\varepsilon$ with sufficiently big imaginary part in the strip $1/2 < \operatorname{Re} z < c/2$. Hence both z_0 and z_1 will be attracting.

Now suppose α^2 lies strictly outside the disk of radius $1/2$ centered at $1/2$. Then $\operatorname{Re}(-1/\alpha^2) \geq -1$, so that

$$\operatorname{Re}\iota_0 + \operatorname{Re}\iota_1 \leq 1,$$

and it cannot be the case that both are attracting, that is, $\operatorname{Re}\iota_i > 1/2$. □

Problem (12-c)**.** Any fixed point z_0 for f is evidently also a fixed point for f^k. If z_0 is attracting (or repelling), show that $\iota(f^k, z_0)$ tends to the limit 1 (or 0) as $k \to \infty$. For a fixed point of multiplicity $m \geq 2$, show that $\iota(f^k, z_0)$ tends to the limit $m/2$.

Proof. If $|\lambda| \neq 1$, then on a chart φ where $\varphi(z_0) = 0$, we locally have that

$$f(z) = \lambda z + a_2 z^2 + \cdots \implies f^k(z) = \lambda^k z + \cdots,$$

that is, the multiplier of f^k at z_0 is λ^k. Then

$$\iota(f^k, z_0) = \frac{1}{1-\lambda^k}$$

which tends to 1 if $|\lambda| < 1$, or to 0 if $|\lambda| > 1$.

If z_0 is a fixed point with multiplicity $m = n + 1 \geq 2$, we consider the normal form

$$f(z) = z + \alpha z^{n+1} + \beta z^{2n+1} + \cdots$$

where $\iota(f, z_0) = \beta/\alpha^2$. We compute the expansion of higher iterates of f. It is straightfoward to see that the higher iterates will still be of the form

$$f^k(z) = z + \alpha_k z^{n+1} + \beta_k z^{2n+1} + \cdots ,$$

and we also see that

$$\begin{cases} \alpha_{k+1} = \alpha_k + \alpha, \\ \beta_{k+1} = \beta_k + \beta + (n+1)\alpha\alpha_k. \end{cases}$$

This implies that $\alpha_k = k\alpha$, as we have already seen, and

$$\beta_{k+1} = \beta_k + \beta + (n+1)k\alpha^2.$$

Therefore

$$\beta_k = k\beta + (n+1)\alpha^2 \frac{k(k-1)}{2}.$$

This implies that

$$\frac{\beta_k}{\alpha_k^2} = \frac{1}{2} \cdot \frac{2k\beta + (n+1)k(k-1)\alpha^2}{k^2\alpha^2} = \frac{1}{2} \cdot \frac{(n+1)k^2\alpha^2 + (2\beta - (n+1)\alpha^2)k}{k^2\alpha^2}$$

whose limit as $k \to \infty$ is $(n+1)/2 = m/2$, that is, the multiplicity over 2. $\square$

Problem (12-d)**.** Verify the generalized fixed point formula of 12.5 in the following special cases:

If $f : \mathbb{T} \to \mathbb{T}$ is a linear map with derivative f' identically equal to α, show that the trace τ of the induced action on the 1-dimensional space of holomorphic 1-forms is equal to α. If f is not the identity map, show that there are $|1 - \alpha|^2$ fixed points, each with index $\iota = 1/(1 - \alpha)$, and conclude that $\sum \iota = 1 - \overline{\tau}$, as required.

Now suppose that S is a compact surface of genus g and that $f : S \to S$ is an involution with k fixed points. Use the Riemann-Hurwitz formula to conclude that the quotient S/f is a surface of genus $\hat{g} = (2 + 2g - k)/4$. For the induced action on the g-dimensional vector space of holomorphic 1-forms, show that $\hat{g}$ of the eigenvalues are equal to $+1$, that the remaining $g - \hat{g}$ are equal to -1, so that the trace τ equals $2\hat{g} - g$. Conclude that $\sum \iota = k/2 = 1 - \overline{\tau}$.

Proof. On a torus, (by Riemann-Roch) the space of holomorphic 1-forms is actually 1-dimensional, generated by dz projected onto $\mathbb{T}$, since translations preserve dz. We also see that the induced map on the pullback $f^* : \Omega(\mathbb{T}) \to \Omega(\mathbb{T})$ acts by multiplication by α:

$$(f^*\omega)_z(v) = \omega_{f(z)}(df_z(v)) = \omega_{f(z)}(\alpha v) = \alpha\omega_{f(z)}(v),$$

so that $f^*\omega = \alpha\omega$. As it is a 1-dimensional complex vector space, the trace τ is also equal to α. (Can we show this without Riemann-Roch?)

We have previously seen that if f is not the identity, it has $|1-\alpha|^2$ fixed points, the solutions of

$$z - f(z) = (1-\alpha)z - c,$$

a map of degree $|1-\alpha|^2$ if $\alpha \neq 1$. As the multiplier of each fixed point is α, the index ι is $1/(1-\alpha)$, so that naturally

$$\sum_{\iota} = |1-\alpha|^2 \cdot \frac{1}{1-\alpha} = \overline{1-\alpha} = 1 - \overline{\alpha} = 1 - \overline{\tau}.$$

For the second part, if $z_0 \in S$ is a fixed point, we see that $df_{z_0} : T_{z_0}S \to T_{z_0}S$ must be such that $(df_{z_0})^2 = \mathrm{Id}$, or more explicitly, the multiplier of f^2 at z_0 must be 1. This implies that the multiplier of f at z_0 is ± 1. Suppose the multiplier is 1, and that locally f is not the identity, so that the multiplicity m of z_0 as a fixed point is ≥ 2, and

$$f(z) = z + a_m z^m + \cdots \implies f^2(z) = z + 2a_m z^m + \cdots$$

but as $f^2 = \mathrm{Id}$, we must have that $a_m = 0$, a contradiction. Hence the multiplier of f at the fixed point z_0 must be -1. We want to find a holomorphic chart φ around z_0 such that locally $f(z) = -z$. (Is this possible?)

Since the problem is local, assume $z_0 = 0$, and let U, V be neighborhoods of 0 such that $U \subseteq V$ and $f(U) \subseteq V$. We may also assume $U = \mathbb{D}_r$ for some $r > 0$, and by conjugating f by z/r, we may assume $r = 1$. Note that for a sufficiently small choice of $r > 0$ initially, we have that f maps $\mathbb{D} \subseteq V$ onto $f(\mathbb{D}) \subseteq V$ biholomorphically. Moreover, by Koebe's one quarter theorem, $\mathbb{D}_{1/4} \subseteq f(\mathbb{D})$.

If we know that the quotient S/f is well defined as a Riemann surface and that the fixed points are exactly the branch points of the quotient $f : S \to S/f$, where locally f is two-to-one, we have that

$$k = 2\chi(S/f) - \chi(S) = 2(2 - 2\hat{g}) + 2g - 2$$

so that $\hat{g} = (2 + 2g - k)/4$.

We know that $\dim \Omega(S) = g$, And the induced pullback map $f^* : \Omega(S) \to \Omega(S)$ is such that $(f^*)^2 = \mathrm{Id}$. We also have the pullback $\pi^* : \Omega(S/f) \to \Omega(S)$. We also consider the pushforward map $\pi_* : \Omega(S) \to \Omega(S/f)$; for $\omega \in \Omega(S)$, let

$$\widetilde{\omega} = \frac{1}{2}(\omega + f^*\omega),$$

so that $\widetilde{\omega}$ is f^*-invariant, and it descends naturally to a holomorphic 1-form on $\Omega(S/f)$. One way to make this explicit is considering $q \in S/f$, $v \in T_q(S/f)$, $p \in \pi^{-1}(q)$ and $u \in T_pS$ such that $df_p(u) = v$. Then we set $(\pi_*\omega)_p(v) = \widetilde{\omega}_p(u)$. This will not depend on the choice of $p \in \pi^{-1}(q)$, and is holomorphic. If q happens to be a branch value, we may apply singularity removability.

Another way of viewing it: if $g : X \to Y$ is a holomorphic finite branched covering, then away from the branch values g is a covering map, so that we may define the pushfoward by the pullback of the local inverses:

$$(g_*\omega)_q = \sum_{p \in g^{-1}(q)} ((g_p^{-1})^*\omega)_p,$$

where g_p^{-1} is a local inverse of g sending q to p. Given that the finite covering will have a group of automorphisms acting on X^0 giving the quotient Y^0, this matches up with the first definition (except up to multiplication by the degree of the covering?).

In any case, if $\alpha \in \Omega(S/f)$, we have that $\pi^*\alpha$ is f^*-invariant, since $\pi \circ f = \pi$. Hence $\widetilde{\pi^*\alpha} = \pi^*\alpha$. It is easy to see then that $\pi_*\pi^*\alpha = \alpha$ (or 2α?). In any case, this allows us to show that π^* is injective, since if $\pi^*\alpha = 0$, then $\alpha = 0$. As seen above, the image of $\Omega(S/f)$ in $\Omega(S)$ is fixed by f^*, giving rise to $\hat{g}$ eigenvalues equal to $+1$. Conversely, if $f^*\omega = \omega$, then $\widetilde{\omega} = \omega$, so that ω will the the pullback of $\pi_*\omega$ (or $\frac{1}{2}\pi_*\omega$?).

More generally, in this specific case of an involution on S, we have the decomposition

$$\omega = \frac{1}{2}(\omega + f^*\omega) + \frac{1}{2}(\omega - f^*\omega)$$

of $\Omega(S)$ as the direct sum of spaces of forms ω such that $f^*\omega = \omega$, and of forms ω such that $f^*\omega = -\omega$. Naturally we then have $g - \hat{g}$ eigenvalues of f^* equal to -1, so that the trace τ is equal to $\hat{g} \cdot 1 + (g - \hat{g}) \cdot (-1) = 2\hat{g} - g$.

Since the multiplier at each fixed point is -1, so that $\iota = 1/2$, we have

$$\sum \iota = \frac{k}{2} = 2\hat{g} - g = 1 - \overline{\tau}.$$

□

Remark: a local treatment of the pushforward of holomorphic forms is also via the following. If $\varphi(z) = z^n$ is the n-degree branched covering of $\mathbb{D}$ over $\mathbb{D}$, and $\omega = z^k dz$, then

$$\varphi_* \omega = \begin{cases} z^{\frac{k+1}{n}-1} dz, & \text{if } k \equiv -1 \mod n; \\ 0, & \text{otherwise.} \end{cases}$$

Remark: If S is a compact Riemann surface of genus $g \geq 2$, whose universal cover is the hyperbolic plane, we have that the group of isometries of S must be a compact Lie Group, whose Lie algebra corresponds to the Killing fields. But because of negative sectional curvature of S, there can be no non-trivial Killing fields on S, so that the group of isometries is discrete, and therefore finite.

If $f : S \to S$ is holomorphic, it will be a branched covering of degree d with b branch points counted with multiplicity, so that

$$b = d\chi(S) - \chi(S) = (d-1)\chi(S) \leq 0,$$

since $\chi(S) = 2 - 2g \leq 0$. Hence we must have that $d = 1$, and f is a biholomorphism from S to S, and therefore preserve the Poincaré metric on S. But because of the finiteness of the group of isometries, f must have finite order, so that for some n, $f^n = \mathrm{Id}$. We may therefore try to generalize the results above, for involutions, to general finite order automorphisms of S. But this would still beget the question of how exactly we know that S/f is a genuine Riemann surface.

Remark: We try to show that S/f is a Riemann surface. Indeed, the covering is well defined away from branch values and branch points, where it is easy to describe a holomorphic chart around points away from branch values. (More explicitly, removing branch values and their preimages, we have a finite group acting without fixed points by holomorphic automorphisms, inducing a holomorphic structure on the quotient.)

Therefore we must understand what happens around the branch points, that is, the fixed points of f. We know that each of them has multiplier -1; it seems intuitive that locally f would act like $z \mapsto -z$ around these fixed points, so that we could construct a chart $z \mapsto z^2$ around the projection of this fixed point. We then consider the following question:

Proposition 12.1. *Let f be a holomorphic germ having a fixed point at 0, so that $f \circ f = \mathrm{Id}$ and $f \neq \mathrm{Id}$. Then there exists a holomorphic conjugation φ, with positive radius of convergence, so that $\varphi \circ f \circ \varphi^{-1}(z) = -z$.*

The not-so-completely precise idea should be: By conjugating by $z \mapsto z/r$, we may assume that f is defined on $\mathbb{D}$. Consider $R_+ = \mathbb{R}_+ \cap \mathbb{D}$ the positive real axis in $\mathbb{D}$, and its image $f(R_+)$. This will cut a smaller disk (for example, $\mathbb{D}_{1/4}$, by Köebe's one quarter theorem) into two parts, one intuitively corresponding to the upper half-disk, and another to the lower. We find a holomorphic map $\psi : \mathbb{D} \cap \mathbb{H}^+$ extending to the boundary which maps $0 \mapsto 0$, $\mathbb{R}_+ \cap \mathbb{D} \to R_+$, and $\mathbb{R}_- \cap \mathbb{D} \to f(R_+)$. We also map the lower half-disk by $z \mapsto f(\psi(-z))$. This resulting map will be holomorphic outside of the real line and continuous on it; by the Schwarz reflection principle, it is holomorphic throughout, giving a conjugacy between f and $z \mapsto -z$.

With this conjugation around the fixed points of $f : S \to S$, we may then construct the charts on S/f around the projection of these fixed points, since the unit disk projects to a disk with a cone point at the origin, mapped 2 to 1 by $z \mapsto z^2$.

13 Most Periodic Orbits Repel

We state the standard version of the implicit function theorem. Consider $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ a C^1 function, where the coordinates in $\mathbb{R}^{n+m}$ are expressed as (x, y). for $(a, b) \in \mathbb{R}^{n+m}$, and

$$F(x, y) = (F^1(x, y), \dots, F^m(x, y)).$$

we have tha Jacobian matrix

$$J_F(a,b) = \begin{bmatrix} \frac{\partial F^1}{\partial x^1}(a,b) & \cdots & \frac{\partial F^1}{\partial x^n}(a,b) & \frac{\partial F^1}{\partial y^1}(a,b) & \cdots & \frac{\partial F^1}{\partial y^m}(a,b) \\ \frac{\partial F^2}{\partial x^1}(a,b) & \cdots & \frac{\partial F^2}{\partial x^n}(a,b) & \frac{\partial F^2}{\partial y^1}(a,b) & \cdots & \frac{\partial F^2}{\partial y^m}(a,b) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(a,b) & \cdots & \frac{\partial F^m}{\partial x^n}(a,b) & \frac{\partial F^m}{\partial y^1}(a,b) & \cdots & \frac{\partial F^m}{\partial y^m}(a,b) \end{bmatrix}$$

or succintly

$$J_F(a,b) = \begin{bmatrix} \frac{\partial F^i}{\partial x^j}(a,b) & \frac{\partial F^i}{\partial y^j}(a,b) \end{bmatrix} = \begin{bmatrix} J_{F,x}(a,b) & J_{F,y}(a,b) \end{bmatrix}.$$

Theorem 13.1. *If $F(a, b) = 0$ and the matrix $J_{F,y}(a, b)$ of partial derivatives with respect to y is invertible, there exists an open subset $U \subseteq \mathbb{R}^n$ containing a and a unique C^1 function $G : U \to \mathbb{R}^n$ such that $G(a) = b$ and, for all $x \in U$, $G(x, F(x)) = 0$. Moreover, the Jacobian matrix of G at $x \in U$ is given by*

$$J_G(x) = -J_{F,y}(x, g(x))^{-1} J_{F,x}(x, g(x)).$$

Therefore, for some $(a', b') \in \mathbb{R}^{m+n}$ such that $F(a', b') = 0$ and a' is sufficiently close to a, then necessarily (a', b') lies in the graph of G, so that $G(a') = b'$.

We can carry over the theorem to the holomorphic case. If $\mathbb{C} \times \mathbb{C}$ has coordinates (z, w), $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a holomorphic function such that $f(a, b) = 0$ and $\frac{\partial f}{\partial w}(a, b) \neq 0$, Then locally around a we may express the solution set $f(z, w) = 0$ of f as the graph of a holomrphic function $g : U \to \mathbb{C}$, where $g(a) = b$.

Suppose now that $f_t : V \times \mathbb{C} \to \mathbb{C}$ is holomorphic, where V is a small neighborhood of 0. Let z_0 be a fixed point of f_0 whose multiplier is $\neq 1$, and $F : V \times \mathbb{C} \to \mathbb{C}$ be given by

$$F(t, z) = f_t(z) - z.$$

We then have that $F(0, z_0) = 0$ and $\frac{\partial F}{\partial z}(0, z_0) = f_0'(z_0) - 1 \neq 0$, so that we may find $U \subseteq V$ small and a unique holomorphic function $g : U \to \mathbb{C}$ such that $g(0) = z_0$ and, for all $t \in U$, $F(t, g(t)) = 0$, that is, $f_t(g(t)) = g(t)$. Explicitly, this means that for a sufficiently small perturbation of $t = 0$, we will find a unique fixed point z_t of f_t "close" to z_0, following along this parametrization $z_t = g(t)$. We may also consider analytic continuations of $z_t = g(t)$ for perturbations $t \in V$. Naturally, the multiplier λ_t of z_t as a fixed point of f_t is a holomorphic function of t as well.

The same reasoning applies if z_0 is periodic point of (prime) period q, whose multiplier $\lambda = (f_0^q)'(z_0)$ is $\lambda \neq 1$. We consider

$$F(t, z) = f_t^q(z) - z,$$

which has q solutions $(0, z_0), (0, f(z_0)), \ldots, (0, f^{q-1}(z_0))$, corresponding to the points in the orbit of z_0. We then find holomorphic functions $g_0, g_1, \ldots, g_{q-1} : U \to \mathbb{C}$ that follow the periodic points so that

$$f_t^q(g_i(t)) = g_i(t),$$

being also a periodic point of period q for z_0. By uniqueness of the functions g_i, we actually have that

$$f_t(g_i(t)) = g_{i+1}(t),$$

since $f_t \circ g_i$ would also satisfy the conditions for g_{i+1}. Hence these points indeed follow the same orbit as z_0, and for small t, we may assume they are all distinct, so that the orbit has prime period q.

What happens more precisely if the multiplier is a root of unity? We have discussed this situation previously, which we repeat here. Suppose $\hat{z}$ is a fixed point for f, whose local expression around $\hat{z}$ is

$$f(z) = \lambda z + az^{m+1} + o(z^{m+1}),$$

so that the multiplier $\lambda = e^{2\pi ip/q}$ is a primitive q-th root of unity. We want to find an explicit local expression of f^k, for curiosity. If

$$f^k(z) = \lambda^k z + a_k z^{m+1} + \cdots,$$

we have that $a_1 = a \neq 0$ and

$$a_{k+1} = \lambda a_k + a(\lambda^k)^{m+1}.$$

Letting $b_k = a_k/\lambda^k$, so that $b_1 = a/\lambda$, we obtain

$$b_{k+1} = b_k + \frac{a}{\lambda}(\lambda^m)^k \implies b_k = \frac{a}{\lambda}\sum_{i=0}^{k-1}(\lambda^m)^i,$$

and therefore $a_k = a\lambda^{k-1}\sum_{i=0}^{k-1}\lambda^{mi}$. Then

$$f^k(z) = \lambda^k z + \frac{a}{\lambda}\left(\sum_{i=0}^{k-1}(\lambda^m)^i\right) z^{m+1} + O(z^{m+2}).$$

We have two different situations, whether $\lambda^m = 1$ or not. Note also that $\lambda^m = 1$ if and only if q divides m:

$$f^k(z) = \begin{cases} \lambda^k z + k\lambda^{k-1}az^{m+1} + \cdots, & \text{if } q \mid m; \\ \lambda^k z + \lambda^{k-1}\dfrac{1-\lambda^{mk}}{1-\lambda^m}az^{m+1} + \cdots, & \text{if } q \not\mid m. \end{cases}$$

The coefficient of z^{m+1} never vanishes in the first case, so that $\hat{z}$ is a fixed point for f^q of multiplicity $m+1$. In the second case, we actually get that the multiplicity of $\hat{z}$ as a fixed point of f^q is greater than $m+1$, as $1-\lambda^{mq} = 0$.

Suppose that $\hat{z}$ is a fixed point of f^q of multiplicity $n+1$. A previous result from the book affirms that n is a multiple of q, as multiplication by λ permutes the n attracting directions of the parabolic fixed point. Moreover, $\hat{z}$ must be a simple fixed point of f^k for all $k < q$, because

$$f^k(z) = \lambda^k z + O(z^2)$$

and $\lambda^k \neq 1$. When we perturb $f = f_0$ generically to some f_t, the $(n+1)$-fold fixed point of f_0^q breaks apart into $n+1$ simple fixed points for f_t^q. We see that $\hat{z}$ must still be perturbed to a simple fixed point $\hat{z}_t$ of f_t; It will be the unique fixed point of f_t in a small neighborhood. The simple fixed point $\hat{z}$ of f_0^k will be preserved to a simple fixed point of f_t^k, which must in fact be the point $\hat{z}_t$.

All the other n simple fixed points of f_t are periodic of period q, and since they cannot be fixed points for f_t^k for $k < q$, the period is exactly q. Hence they are arranged into n/q orbits of period exactly q.

Possibly a similar analysis of perturbation can be made when $\hat{z}$ is periodic of period l with multiplier a primitive q-th root of unity, but with more details.

(Third edition: I have not seen the studies on the residue itératif, or the definitions on parabolic attracting and parabolic repelling fixed points, but may look into these at a later time. These give better descriptions as to whether you get attracting or repelling fixed points when you perturb a parabolic fixed point.)

14 Repelling Cycles are Dense in J

Remark on Proof 14.1, following Julia: there are possibly many ways of strengthening the statement that the repelling periodic orbits are dense in the Julia set. One way of doing this through the proof is being able to show that, for any finite collection of open sets $U_1, \ldots, U_m$ such that each U_i intersects $J(f)$, there exists a repelling orbit $\mathcal{O}$ passing through all of the U_i. To see this, it is enough to consider more preimages of $z_r \in U_1$ passing through the open sets $U_i, i \geq 2$ before coming back to the neighborhood N_0. This is also a stronger statement than just topological transitivity of f on $J(f)$.

Comment on Proof of 14.4: To be more explicit in the proof: from the arguments, it is straightforward to see that indeed some subsequence of interated inverse maps $\{f^{-n_i}\}$ defined on $\Delta \cap V$ converge locally uniformly to the identity on $\Delta \cap V$. To upgrade this convergence to be on all of V, we check that the orbits of the points in V under $\{f^{-n_i}\}$ avoid the central part of Δ; otherwise we would have a point in this central part of Δ mapping to V under some f^{n_i}, which cannot happen as Δ is a rotation domain, preserving inner disks. The rest of the proof follows.

15 Herman Rings

Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an arbitrary orientation-preserving continuous function. We show that the rotation number is well defined and independent of both the choice of lift F and the choice of $t_0 \in \mathbb{R}$. Naturally, if F is a given lift, we have the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{f} & \mathbb{R}/\mathbb{Z} \end{array}$$

so that $p(F(t)) = f(p(t))$. Note that for $n \in \mathbb{N}$, F^n is a lift of f^n by induction, and if f is a homeomorphism, this is true for $n \in \mathbb{Z}$.

Now assume $\widetilde{F}$ is another arbitrary lift of f. As p is an additive group homomorphism, we have that

$$p(F(t) - \widetilde{F}(t)) = p(F(t)) - p(\widetilde{F}(t)) = f(p(t)) - f(p(t)) = 0,$$

so that $F(t) - \widetilde{F}(t)$ must be constantly equal to some integer by continuity. This concludes that any lift $\widetilde{F}$ of f is equal to F up to some integer additive constant.

Note that $F(t+1)$ is also a lift of f, since $p(F(t+1)) = f(p(t+1)) = f(p(t))$, so that

$$F(t+1) = F(t) + m,$$

for some $m \in \mathbb{Z}$. Inductively, for $l \in \mathbb{N}$, $F(t+l) = F(t) + lm$, and then $F^2(t+1) = F(F(t)+m) = F^2(t) + m^2$. This generalizes to

$$F^n(t+1) = F^n(t) + m^n$$

by induction as well.

If f preserves orientation, then $m \geq 0$, And if f is a homeomorphism, must have $m = 1$ and

$$F(t+1) = F(t) + 1,$$

otherwise for some $s \in (t, t+1)$ we would have $F(s) = F(t)+1$, contradicting injectivity of f. Furthermore, $F^n(t+1) = F^n(t) + 1$ for $n \in \mathbb{Z}$.

Considering only the case where f is a homeomorphism, we define

$$G(t) = F(t) - t,$$

where from the above G is a periodic function of period 1, hence bounded by some $M \in \mathbb{R}$. Therefore

$$\frac{F^n(t) - t}{n} = \frac{1}{n}\sum_{i=0}^{n-1}(F^{i+1}(t) - F^i(t)) = \frac{1}{n}\sum_{i=0}^{n-1} G(F^i(t))$$

is also bounded by M. More generally, for $n_2 > n_1 \in \mathbb{N}$, we have

$$\frac{F^{n_2}(t) - F^{n_1}(t)}{n_2 - n_1} = \frac{1}{n_2 - n_1} \sum_{i=n_1}^{n_2-1} (F^{i+1}(t) - F^i(t)) = \frac{\sum_{i=n_1}^{n_2-1} G(F^i(t))}{n_2 - n_1} \leq M.$$

In order to consider whether the limit $\frac{F^n(t)}{n}$ exists for some $t \in \mathbb{R}$, we see that if it exists for a given t, then it exists and is equal for all $t + m$, $m \in \mathbb{Z}$, given that

$$\frac{F^n(t+m)}{n} = \frac{F^n(t) + m}{n} = \frac{F^n(t)}{n} + \frac{m}{n},$$

and $m/n \to 0$. Now, for $t_1, t_2 \in [0, 1)$ such that $t_1 < t_2$, as F is monotone increasing, we have

$$F^n(t_2) - F^n(t_1) \leq F^n(1) - F^n(0) = 1.$$

This implies that

$$\frac{F^n(t_2) - F^n(t_1)}{n} \to 0, \text{ as } n \to \infty,$$

so that if the limit exists for t_1, it also exists for t_2. Hence we only need to prove that the limit exists for one $t \in \mathbb{R}$.

I will not prove here that the limit exists: the proof can be found in Devaney's *An Introduction to Chaotic Dynamical Systems.*

It is also straightforward to see that

$$\text{rot}(f^k) = k \cdot \text{rot}(f^k),$$

since

$$\text{Rot}(F^k) = \lim_{n\to\infty} \frac{F^{kn}(t_0)}{n} = k \cdot \lim_{n\to\infty} \frac{F^{nk}(t_0)}{nk} = k \cdot \text{rot}(F).$$

Remark on Lemma 15.7: The fact that ∂U has exactly two components, both of which are connected, stem from the fact that U itself is conformally equivalent to an annulus where f acts as an irrational rotatio, thereby preserving concenctric circles. If the annulus is $A_R = \{1 < |z| < R\}$ and φ is the conformal equivalence, we then have the subsets

$$C_{a,b} = \{z \in U : a < |\varphi(z)| < b\}$$

of U, and we may consider the nested intersections

$$\bigcap_{n\geq 1} \overline{C_{1,1+1/n}}, \quad \bigcap_{n\geq 1} \overline{C_{R-1/n,R}}$$

accumulating on the two components of ∂U, being compact connected sets. Moreover, we can see that they cannot be single points, otherwise U would be a punctured disk or punctured complex plane (?).

Problem (15-a. No polynomial Herman rings)**.** Using the maximum modulus principle, show that no polynomial can have a Herman ring.

Proof. Suppose U is a Herman ring for a polynomial $f : \mathbb{C} \to \mathbb{C}$, such that $\mathbb{C} \backslash U$ has two components A and B, where A is compact and B is unbounded, containing a neighborhood of infinity. Then $\partial A \subseteq \partial U$, and by the maximum modulus principle,

$$\max_{z \in A} |f(z)| = \max_{z \in \partial A} |f(z)|.$$

But for $z \in \partial U$, its orbit is bounded, being contained in ∂U since $f(U) = U$ and therefore $f(\overline{U}) \subseteq \overline{f(U)} = \overline{U}$. Moreover,

$$A \subseteq A \cup \operatorname{int}\{z \in U \colon 1 < |\varphi(z)| < r\}$$

for all $r > 1$, where the orbit is still bounded. This implies not only that $\operatorname{int} A \subseteq F(f)$, but that $A \subseteq F(f)$. But this gives a contradiction, since $\partial A \subseteq \partial U$ must be contained in $J(f)$. □

Problem (15-b. Symmetry of Blaschke products)**.** For any Blaschke product $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ show that the Julia set is invariant under the inversion $z \mapsto 1/\overline{z}$. Show that z is a critical point of f if and only if $1/\overline{z}$ is a critical point, and show that z is a zero of f if and only if $1/\overline{z}$ is a pole.

Proof. Recall that a Blaschke product is of the form

$$f(z) = e^{2\pi i \lambda} \beta_{a_1}(z) \cdots \beta_{a_k}(z),$$

where

$$\beta_a(z) = \frac{1 - \overline{a}}{1 - a} \cdot \frac{z - a}{1 - \overline{a}z}.$$

(In this case, the factor $(1 - \overline{a})/(1 - a)$ is included to ensure that the point $z = 1$ is fixed.) Note that

$$\beta_a(1/\overline{z}) = \frac{1 - \overline{a}}{1 - a} \cdot \frac{1/\overline{z} - a}{1 - \overline{a}/\overline{z}} = \frac{1 - \overline{a}}{1 - a} \cdot \frac{1 - \overline{z}a}{\overline{z} - \overline{a}},$$

so that

$$\frac{1}{\overline{\beta_a(1/\overline{z})}} = \frac{1 - \overline{a}}{1 - a} \cdot \frac{z - a}{1 - \overline{a}z} = \beta_a(z).$$

Since β_a is conjugate to itself by $z \mapsto 1/\overline{z}$, it must map its Julia set to itself via the inversion. In general, if $g \circ f \circ g^{-1} = f$, then g is a symmetry of $J(f)$.

Moreover, since g is a diffeomorphism (or in this case an anticonformal isomorphism), that it preserves critical points is straightfoward from the chain rule on $f \circ g = g \circ f$. And since g interchanges 0 and ∞, the last part also follows easily. □

Problem (15-c. Proper self-maps of $\mathbb{D}$)**.** A holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is said to be proper if the inverse image of any compact subset of $\mathbb{D}$ is compact. Show that any proper holomorphic map from $\mathbb{D}$ onto itself can be expressed uniquely as a Blaschke product, with $a_j \in \mathbb{D}$.

Proof. Uniqueness follows easily from the description of the Blasckhe products, since $a_1, \ldots, a_k\} = f^{-1}(0)$ and $e^{2\pi i \lambda} = f(1)$. As f is a proper holomorphic map (being automatically surjective), it is a global holomorphic branched covering, having a well defined degree d, as the number of preimages of a non-critical value. Suppose $a \in f^{-1}(0)$, and multiplty f by $(\beta_a)^{-1}$. This will reduce the degree of f by one, and inductively, we may recover f as a Blaschke product. □

Problem (15-d. Computing rotation numbers)**.** (1) Show that the rotation number $\mathrm{rot}(f)$ can be deduced directly from the cyclic order relations of a single orbit, in a form convenient for computer calculations, as follows. Choose representatives $t_i \in [0, 1)$ for the elements of the orbit of zero, so that $t_i \equiv f^i(0) \mod \mathbb{Z}$. If we exclude the trivial case $t_1 = 0$, then t_1 cuts $[0, 1)$ into two disjoint intervals $I_1 = [0, t_1)$ and $I_0 = [t_1, 1)$. Define a sequence of bits $(b_2, b_3, b_4, \ldots)$ by the requirement that $t_n \in I_{b_n}$. If F is the unique lift with $F(0) = t_1$, show that

$$\mathrm{Rot}(F) = \lim_{n\to\infty} \frac{b_2 + b_3 + \cdots + b_n}{n}.$$

(2) Furthermore, if a second such map f' has bit sequence $(b'_2, b'_3, b'_4 \ldots)$, and if

$$(b_2, b_3, b_4, \ldots) < (b'_2, b'_3, b'_4, \ldots)$$

using the lexicographic order for bit sequences, show that

$$\mathrm{Rot}(F) \leq \mathrm{Rot}(F').$$

Proof. We may consider $b_1 = 0$. Let k_n be the interger uniquely defined integer such that $t_n \in [k_n, k_n + 1)$, where $k_1 = 0$. We know that $F(0) = t_1$

and $F(t+1) = F(t)+1$, so that F mas the interval $[0,1]$ homeomorphically onto $[t_1, t_1+1]$. This shows that if $b_2 = 0$, then $t_2 = F^2(0) \in [t_1, 1)$ and $k_2 = 0$, and if $b_2 = 1$, then $F^2(0) \in [1, t_1+1)$ and $k_2 = 1$. In essence, k_n represents how many rotations f has made on the circle over n iterates.

Suppose then $t_n \in [k_n, k_n+1)$, where $t_n \in [k_n, k_n + t_1)$ or $t_n \in [k_n + t_1, k_n+1)$, according to whether $b_n = 1$ or $b_n = 0$, respectively. Note that $F[k_n, k_n+1) = [k_n + t_1, k_n + t_1 + 1)$, monotonically. This image is then partitioned into

$$F[k_n, k_n+1) = [k_n+t_1, k_n+1+t_1) = [k_n+t_1, k_n+1) \cup [k_n+1, k_n+1+t_1),$$

corresponding to the intervals I_0 and I_1 on the circle respectively. This means that if $b_{n+1} = 0$, we must have that $t_{n+1} \in [k_n + t_1, k_n+1)$, and if $b_{n+1} = 1$, then $t_{n+1} \in [k_n+1, k_n+1+t_1)$. Consequently,

$$\begin{cases} b_n = 0 & \implies k_{n+1} = k_n, \\ b_n = 1 & \implies k_{n+1} = k_n + 1. \end{cases}$$

In other words, $k_{n+1} = k_n + b_{n+1}$. This in fact shows that

$$k_n = b_1 + b_2 + \cdots + b_n.$$

Moreover, since

$$\frac{k_n}{n} \le \frac{F^n(0)}{n} < \frac{k_n}{n} + \frac{1}{n},$$

we have that the limit $\lim k_n/n$ exists and is equal to $\mathrm{Rot}(F)$.

For the second part, suppose that for $n < N$, we have $k_n = k'_n$, so that $F^n(0), F'^n(0) \in [k_n, k_n+1)$, and for $n = N$, we have the first "disagreement" between the sequences, so that $b_N = 0$, $b'_N = 1$. This implies that

$$k_N \le F^N(0) < k_N + 1 \le F'^N(0) < k_N + 2,$$

and more precisely, that

$$k_N + t_1 \le F^N(0) < k_N + 1 \le F'^N(0) < k_N + 1 + t_1.$$

Now, considering that F and F' are monotone increasing, we have

$$\begin{aligned} F^N(k_N + t_1) &\le F^{2N}(0) < F^N(k_N+1) \\ \implies F^N(t_1) + k_N &\le F^{2N}(0) < F^N(0) + k_N + 1 < 2(k_N+1), \end{aligned}$$

so that inductively $F^{jN}(0) < j(k_N+1)$ for $j \ge 1$. Similarly,

$$\begin{aligned} F'^N(k_N+1) &\le F'^{2N}(0) < F'^N(k_n+1+t_1) \\ \implies F'^N(0) + k_N + 1 &\le F'^{2N}(0) < F'^N(t_1) + k_N + 1, \end{aligned}$$

so that inductively $j(k_N+1) \le F'^{jN}(0)$. With this, we naturally have that

$$\frac{F^{jN}(0)}{jN} \le \frac{k_N+1}{N} \le \frac{F'^{jN}(0)}{jN},$$

and $\mathrm{Rot}(F) \le \mathrm{Rot}(F')$. □

Can every bit sequence be realized? Surely not, because the limit of k_n/n has to exist, and I don't believe all such sequences have a limit. Moreover, they need to be realizable so that they respect the above inequality. In a way, the initial behavior of the bit sequence has to constrain the future behavior, only making it more precise.

In fact, consider the following. Since $k_N \le F^N(0) < k_N+1$, we get

$$\begin{aligned} & F^M(k_N) \le F^{M+N}(0) < F^M(k_N+1) \\ \implies & F^M(0)+k_N \le F^{M+N}(0) < F^M(0)+k_N+1 \\ \implies & k_M+k_N \le F^{M+N}(0) < k_M+k_N+2, \end{aligned}$$

so that

$$k_M+k_N \le k_{M+N} \le k_M+k_N+1,$$

and the sequence of the k_N is "almost additive". In fact, the sequence $(k_n+1)_{n\in\mathbb{N}}$, by the above inequality on the right hand side, is subadditive. This is sufficient to show that the limit

$$\lim_{n\to\infty} \frac{k_n+1}{n} = \lim_{n\to\infty} \frac{k_n}{n}$$

exists, being defined as the rotation number of F.

This still does not answer the question of which bit sequences are realizable, but gives a slightly clearer picture.

16 The Sullivan Classification of Fatou Components

Problem (16-a. Limits of iterates)**.** Give a sharper formulation of the defining property of the Fatou set $\hat{\mathbb{C}} \setminus J$ for a rational function as follows. If V is a connected open subset of $\hat{\mathbb{C}} \setminus J$, show that the set of all limits of succesive iterates $f^n|_V$ as $n \to \infty$ is either (1) a finite set of constant maps from V into an attracting or parabolic periodic orbit, or (2) a compact one-parameter family of maps, consisting of all compositions $R_\theta \circ f^k|_V$, with $k_0 \le k < k+p$. Here f^{k_0} is to be some fixed iterate with values in a rotation domain belonging to a cycle of rotation domains of period p, and R_θ is the rotation of this domain through angle θ.

Proof. If $V \subseteq \hat{\mathbb{C}} \setminus J = F$ is open and connected, then there exists a unique Fatou component U containing V. We know that all Fatou components are preperiodic. Suppose $f^k(V)$ is mapped into some periodic component corresponding to an immediate basin of an attracting periodic orbit or the immediate basin of a petal of a parabolic periodic orbit, and k_0 is minimal with this property. Then if the period is p, naturally f^{k_0+np+r} converges locally uniformly on V to an attracting periodic point or parabolic periodic point of the orbit as $n \to \infty$, and $0 \leq r < p$. All limits of iterates of f must be of this form.

If $f^{k_0}(V)$ is mapped into some cycle of rotation domain (with irrational rotation number α), We know that orbit $n\alpha$ in the circle is dense, producing this one parameter family R_θ, and we can map into any of the components of this cycle of rotation domains by f^k, $k_0 \leq k < k + p$. □

Problem (16-b. Counting Components)**.** (1) If a quadratic polynomial map has either an attracting fixed point or a parabolic fixed point of multiplier $\lambda = 1$, show that there is only one bounded Fatou component.

(2) If it has an attracting cycle of period 2, show that there are three bounded components which map according to the pattern $U_1 \leftrightarrow U_0 \leftarrow U_1'$ and that the remaining bounded components are iterated preimages of U_1' where each set $f^{-n}(U_1')$ is made up of 2^n distinct components.

(3) What is the corresponding description for a cycle of attracting or parabolic basins with period p, or for the case of a Siegel fixed point?

Proof. (1) Recall that if f is a quadratic polynomial, then ∞ is a superattracting fixed point, in which the multiplicity of ∞ as a critical point is 1. There is therefore only one other finite critical point c. If f has an attracting fixed point or a parabolic fixed point p_0, then necessarily c is contained in the immediate basin of attraction of p_0. In the case of a parabolic fixed point, p_0 is a fixed point of multiplicity 2, since $f(z) - z$ has at most two roots. (Another approach is to recall that a rational map of degree d has exactly $d + 1$ fixed points, counted with multiplicity.) Therefore p_0 has exactly one attracting direction and one repelling direction.

Let $K = \mathbb{C} \setminus \mathcal{A}(\infty)$ be the filled Julia set, such that K is fully invariant by the maximum modulus principle; in fact $\operatorname{int} K \subseteq F(f)$. If U is a bounded Fatou component, then $U \subseteq \operatorname{int} K$, and $f^n(U) \subseteq \operatorname{int} K$. As U must be preperiodic, in must map onto some cycle of Fatou components, corresponding to either a cycle of immediate basins of attraction for an attracting or parabolic periodic orbit, or a cycle of Siegel disks, since Herman rings cannot occur for polynomial maps.

However, since the unique finite critical point c is already in the immediate basin of p_0, no other cycle of attracting or parabolic basins can occur, nor a cycle of Sigel disks. Hence U is mapped into $\mathcal{A}(p_0)$.

Due to the existence of the critical point c in the basin, $f|_{\mathcal{A}(p_0)} : \mathcal{A}(p_0) \to \mathcal{A}(p_0)$ is a proper holomorphic map of degree 2. Hence $\mathcal{A}(p_0)$ must in fact be fully invariant, so that $U = \mathcal{A}(p_0)$. So we prove that f has exactly one bounded Fatou component $\mathcal{A}(p_0)$ and exactly one unbouded Fatou component $\mathcal{A}(\infty)$.

Moreover, $\mathcal{A}(\infty)$ is simply connected in $\hat{\mathbb{C}}$ since K and J are connected (as the finite critical point does not converge to ∞), so that $\mathcal{A}(p_0)$ is also simply connected. The Julia set is the common boundary of $\mathcal{A}(p_0)$ and $\mathcal{A}(\infty)$.

(2) Now assume that there exists an attracting cycle p_0, p_1 of period 2. Let U_i be the Fatou component containg p_i, where we assume that the finite critical point $c \in U_0$. As the orbit of the critical point is bounded, $\mathcal{A}(\infty)$ is connected and simply connected in $\hat{\mathbb{C}}$, and K and J are connected. This implies that each bounded Fatou component is also simply connected, for otherwise J would be disconnected.

As $c \in U_0$, the proper map $f|_{U_0} : U_0 \to U_1$ has degree 2, so that $f^{-1}(U_1)$ has exactly one component, being U_0. The map $f : U_1 \to U_0$ has degree 1, so that $f^{-1}(U_0)$ consists of exactly two simply connected components U_1 and U_1' mapping conformally isomorphically onto U_0 by f.

If U is bounded Fatou component of f, then necessarily some iterate $f^n(U)$ is mapped into U_0, as by the same arguments as above it cannot be mapped to some other attracting or parabolic cycle, or cycle of rotation domains. Therefore if $U \neq U_0, U_1$, then U is a component of $f^{-n}(U_1')$. All of these components must be simply connected and be mapped conformally isomorphically under f^n to U_0, so that there are exactly 2^n such components given that the degree of f^n is 2^n.

(3) If there is a cycle

$$U_0 \to U_1 \to \cdots \to U_{p-1}$$

of attracting or parabolic basins of period p, where we may assume that the critical point c is in U_0, then the map $f|_{U_0} : U_0 \to U_1$ has degree 2, and all the other maps $f|_{U_i} : U_i \to U_{i+1}$ are of degree 1. Again, all Fatou components will be simply connected in $\hat{\mathbb{C}}$. We have that the preimage of U_1 is the single component U_0, while each other component U_i has exactly two simply connected preimages U_{i-1} and U'_{i-1}. All other bounded Fatou components must map into some U_i, and there are 2^n components of $f^{-n}(U'_i)$.

□

Problem (16-c. Wandering domains)**.** Show that the transcendental map $f(z) = z + \sin(2\pi z)$ has one family of wandering components $\{U_n\}$ with $f(U_n) = U_{n+1}$ and one family $\{V_n\}$ with $f(V_n) = V_{n-1}$. Describe the Fatou set for the corresponding map of the cylinder $\mathbb{C}/\mathbb{Z}$.

Proof. Note that for each $n/2 \in \frac{1}{2}\mathbb{Z}$, we have $f(n/2) = n/2$, with multiplier $f'(n) = 1 \pm 2\pi$, being a repelling fixed point. If we let $z = k + 1/4$ for $k \in \mathbb{Z}$, we get the orbit

$$\cdots \mapsto -k + \frac{1}{4} \mapsto \cdots \mapsto -1 + \frac{1}{4} \mapsto \frac{1}{4} \mapsto 1 + \frac{1}{4} \mapsto \cdots \mapsto k + \frac{1}{4} \mapsto \cdots$$

and analogously, for $z = k + 3/4$, we get

$$\cdots \leftarrow -1 + \frac{3}{4} \leftarrow \frac{3}{4} \leftarrow 1 + \frac{3}{4} \leftarrow 2 + \frac{3}{4} \leftarrow \cdots .$$

Note that the multiplier at each of these points is equal to 1.

The map f admits several symmetries. We have $f(z + 1) = f(z) + 1$, so that f is conjugate to itself by the translation $z \mapsto z + 1$. This shows that integer translations are symmetries of the Fatou and Julia sets. Since $\overline{\sin(2\pi \overline{z})} = \sin(2\pi z)$, we have that $\overline{f(\overline{z})} = f(z)$, and conjugation is another symmetry. Finally, $-f(-z) = f(z)$, so that $z \mapsto -z$ is a symmetry, and also the reflection about the imaginary axis $z \mapsto -\overline{z}$, along with reflection over all

We want to understand the relationship between 1/4 and the Fatou set. For z close to 1/4,

$$f(z) - f(1/4) = z - 1/4 + \sin(2\pi z) - 1,$$

where the difference $\sin(2\pi z) - 1$ is of the order of $(z - 1/4)^2$, since

$$\sin(2\pi z) = 1 - \frac{(2\pi)^2}{2}(z - 1/4)^2 + \frac{(2\pi)^4}{24}(z - 1/4)^4 - \cdots$$

If $z = x \in \mathbb{R}$ and x is close to 1/4 but $x \neq 1/4$, we get that

$$(x - 1/4) - \frac{(2\pi)^2}{2}(x - 1/4)^2 < f(x) - f(1/4) < x - 1/4,$$

so that if $x > 1/4$, then the distance $d_n = |f^n(x) - f^n(1/4)|$ is monotone decreasing, and if $x < 1/4$, the distance is monotone increasing for small times n.

Since $f(z+1) = f(z) + 1$, we have an induced holomorphic map on the cylinder $\hat{f} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$. For this map, $1/4$ is a parabolic fixed point with multiplier 1 and multiplicity 2, because locally

$$\hat{f}(z) = \frac{1}{4} + (z - 1/4) - \frac{(2\pi)^2}{2!}(z-1/4)^2 + \frac{(4\pi)^4}{4!}(z-1/4)^4 - \cdots ,$$

around $1/4 \mod \mathbb{Z}$, since $3/4 \equiv 1/4 \mod \mathbb{Z}$. Conjugating by a translation $z \mapsto z - 1/4$, we get that locally

$$\hat{f}(z) = z - \frac{(2\pi)^2}{2!}z^2 + \frac{(4\pi)^4}{4!}z^4 - \cdots$$

so that there is a unique attracting direction and an unique repelling direction. These are exactly the positive and negative real axes that we have described above, whether $x > 1/4$ or $x < 1/4$. This shows that we have an attracting petal $\mathcal{P}$ at this fixed point, and we may lift it to the map $f : \mathbb{C} \to \mathbb{C}$. If $\pi^{-1}(\mathcal{P})$ is the preimage of this petal under the projection $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$, we see that $\pi^{-1}(\mathcal{P})$ is forward invariant under f, and as its complement has at least three points (the repelling fixed points at $k/2$, $k \in \mathbb{Z}$), by normality $\pi^{-1}(\mathcal{P})$ is contained in the Fatou set.

We can also consider the parabolic fixed point at $3/4$ on $\mathbb{C}/\mathbb{Z}$, admiting an attracting petal $\mathcal{Q}$. In fact, one is mapped into the other by the symmetry $z \mapsto 1 - \overline{z}$, which is the reflection about the axis $\operatorname{Re} z = 1/2$ descended to $\mathbb{C}/\mathbb{Z}$.

This lift $\pi^{-1}(\mathcal{P})$ will have components containing petals "adjacent" to the points $k + 1/4$ for $k \in \mathbb{Z}$, and they will be mapped to one another by f. We must show that these components are distinct.

If the component U_n containing $n+1/4+\varepsilon_0$ for some fixed choice of $\varepsilon_0 > 0$ small were the same as the component $U_{n'}$ for $n' > n$, then we may find a smooth path connecting $n + 1/4 + \varepsilon_0$ and $n' + 1/4 + \varepsilon_0$ within the Fatou set. By the symmetry of F under conjugation, we may also assume that this path is contained in the upper half-plane $\mathbb{H}$ except at the endpoints. Translating this path under the symmetry $z \mapsto z+1$, we find a path within F connecting $(n+1) + 1/4 + \varepsilon_0$ and $(n'+1) + 1/4 + \varepsilon_0$. But we must see that since both are contained in $\mathbb{H}$ except at the endpoints, they must intersect. This shows that $U_n = U_{n+1}$, and we have a path within $F \cap \mathbb{H}$ connecting $n + 1/4 + \varepsilon_0$ to $(n+1) + 1/4 + \varepsilon_0$.

If γ is such a smooth path, then $\overline{\gamma}$ also connects the endpoints, but in F intersected with the lower half-plane. Taking small neighborhoods of these paths, we get an annulus within F containing the endpoints.

However, this sequence of annuli between the components U_n must map by the reflection $z \mapsto 1 - \overline{z}$ to a sequence of annuli between the components V_n, and they must be disjoint, which cannot happen. This reasoning implies that each component U_n is distinct, along the with the distinct components V_n. □

What more can be said about the Fatou said for $\mathbb{C}/\mathbb{Z}$, along with the components and the symmetries described?

Problem (16-d. A Baker domain)**.** Show that the map

$$f(z) = z + e^z - 1$$

has a fully invariant Baker domain $U = f^{-1}(U)$. In particular, show that all critical values belong to the half-plane $\operatorname{Re} z < 0$ and that all orbits $\{z_j\}$ in this half-plane satisfy $\lim_{j\to\infty} \operatorname{Re}(z_j) = -\infty$. Show that there is an associated map of the cylinder $\mathbb{C}/2\pi i\mathbb{Z}$.

Proof. The map f has the symmetry $f(z + 2\pi i) = f(z) + 2\pi i$, so that translation by $2\pi i$ is also a symmetry of F and J. Moreover, the map f descends to a holomorphic map of the cylinder $\mathbb{C}/2\pi i\mathbb{Z}$. The critical points of f are of the form $(2k+1)\pi i$, for $k \in \mathbb{Z}$, so that the critical values are $-2+(2k+1)\pi i$, having real part -2.

If $\operatorname{Re} z < 0$, then $|e^z| = e^{\operatorname{Re} z} < 1$, so that the real part of $f(z)$ is

$$\operatorname{Re} f(z) = \operatorname{Re} z + \operatorname{Re} e^z - 1 < \operatorname{Re} z + |e^z| - 1 < \operatorname{Re} z.$$

This means that the left half-plane P for $\operatorname{Re} z < 0$ is forwards invariant under f, that is, $f(P) \subseteq P$. Moreover, it must be contained in the Fatou set, since orbits in it omit at least three points (in particular all points with non-negative real part). Let U be the Fatou component of f which contains this half-plane P. Note that $U \neq \mathbb{C}$, since f has fixed points at $z = 2\pi ki$ for $k \in \mathbb{Z}$, with multipliers equal to 2, hence repelling. Therefore U is a connected hyperbolic Riemann surface.

Note that the orbit of -1 diverges to $-\infty$, so that by the classification of the dynamics on a hyperbolic surface, no other orbit has an accumulation point, and they in fact must follow the same orbit under the Poincaré distance. Moreover, as for all $\varepsilon > 0$ such that $\operatorname{Re} z < -\varepsilon$ we also have $\operatorname{Re} f(z) < \operatorname{Re} z + e^{-\varepsilon} - 1$, where again $\operatorname{Re} f(z) < \operatorname{Re} z < \varepsilon$, we have that

$$\operatorname{Re} f^n(z) < \operatorname{Re} z - n(1 - e^{-\varepsilon}),$$

so that $\operatorname{Re} f^n(z) \to -\infty$. We only need to prove that U is fully invariant. Suppose that V were a component of $f^{-1}(U)$ not equal to U, so that V is contained in the open right half-plane $\operatorname{Re} z > 0$. This means that for some $n \in \mathbb{N}$ and $z \in V$, $\operatorname{Re} f^n(z) < 10$, say. If V were bounded, by the maximum modulus principle of harmonic functions, The minimum of $\operatorname{Re} f^n(z)$ on $\overline{V}$ is attained on the boundary, so that for some $z \in \partial V \subseteq J$ we would have $f^n(z) \in U$, a contradiction. Hence $f^{-1}(U)$ can have no bounded component.

The map f also admits the symmetry $\overline{f(\overline{z})} = f(z)$, so that the Julia and Fatou sets are symmetric about the real axis. This implies that they are symmetric with respect to reflections about the lines $\operatorname{Im} z = (2k+1)\pi i$, and this symmetry descends to $\mathbb{C}/2\pi i\mathbb{Z}$.

We also see that U must contain all points of the form $\operatorname{Im} z = (2k+1)\pi i$, since $e^z = -e^{\operatorname{Re} z}$, so that the imaginary part of z is preserved under f and its real part is stricly decreasing by at least 1 each iteration. This all such points eventually fall into U, and naturally there is a path connecting them to U through the Fatou set, along this line.

In fact, if $\operatorname{Re} f(z) < 0$ and $z = x + iy$, then we must have that

$$\cos y < (1-x)e^{-x},$$

where the right hand side tends to 0 as $\operatorname{Re} z \to 0$. The function $(1-x)e^{-x}$ has a point of global minimum at $x = 2$, where it is equal to $-e^{-2}$. Therefore, whenever $\cos y < -e^{-2}$, we have that z is mapped to the left half-plane P; more explicitly, There is a small constant $c > 0$ such that the countable collection of horizontal strips

$$\{z \in \mathbb{C} \colon \operatorname{Im} z \in (2k\pi + \pi/2 + c, 2k\pi + 3\pi/2 - c)\}$$

is contained in the $f^{-1}(P)$, hence in the Fatou set. By connectedness, they are also contained in U. We also have that each horizontal line segment

$$\{z \in \mathbb{C} \colon \operatorname{Re} z \geq 0, \operatorname{Im} z = 2k\pi,\ k \in \mathbb{Z}\}$$

is disjoint from U, since it consists either of the fixed point at $2k\pi i$, which is repelling, or points wich diverge to $\operatorname{Re} z \to +\infty$. We can also see that U contains the closed left half-plane $\operatorname{Re} z \leq 0$ except for the fixed points $z = 2k\pi i$, $k \in \mathbb{Z}$.

If V were another unbounded component of $f^{-1}(U)$, then necessarily it must be unbounded in the direction of $\operatorname{Re} z \to +\infty$, and it must be contained in a small neighborhood of a half-strip

$$\{z \in \mathbb{C} \colon \operatorname{Re} z > 0,\ \operatorname{Im} z \in (2k\pi - \pi/2, 2k\pi + \pi/2),\ k \in \mathbb{Z}\}.$$

(I have not been able to complete the proof: One could attempt to prove that U is dense in $\mathbb{C}$.) □

17 Prime Ends and Local Connectivity

It seems as though we can describe prime ends as the inverse limit of an inverse system much like the ends of a topological surface (or space). Given a crosscut A in U, we have the two components $\mathrm{Comp}(U \setminus A) = \{N_1, N_2\}$, and if A' is another cross cut which does not intersect A, then we may assume that $N_1' \subset N_1$, but $N_2' \supset N_2$. Maybe the inverse system should be of crosscut neighborhoods.

In fact, it seems unlikely that we would be able to do so; by mimicking an inverse limit construction, one would expect to obtain the prime ends as some for of Cantor set or totally disconnected set. However, for $\mathbb{D}$, its prime ends coincide with $\partial\mathbb{D} = S^1$, a continuum.

Small comment: After theorem 17.16, the book claims that the boundary of a simply connected set is always connected (by Problem 5-b). However, this surely must require more restrictions of the simply connected set; consider for example an infinite strip in $\mathbb{C}$, which has two boundary components as straight lines. Maybe it refers to simply connected sets in $\hat{\mathbb{C}}$.

We also explore (in the future) a little more a concept introduced in Epstein's paper *Prime Ends*, that of principal points (or the principal set or a prime end). One way to possibly formalize is idea is that, for a prime end $\mathcal{E}$, we may consider all equivalent fundamental chains $(N_j)_j$ that give rise to this prime end, and the limit set of the crosscuts A_j, since their diameters tend to 0. (The crosscuts may not convergence to a single point, but a subsequence of them should.) Intuitively, these principal points should also be accumulation points of all rays converging to the prime end.

18 Polynomial Dynamics: External Rays

For the sake of repetition: for a monic polynomial f of degree $d \geq 2$, the fact that all of the bounded Fatou components $U \subseteq \operatorname{int} K(f)$ are simply connected stems from the maximum modulus principle, by taking a simple closed curve Γ in U bounding an open domain V, where we may conclude that $V \subset U$.

Theorem 18.5 is the following:

Let f be a polynomial map with connected and locally connected Julia set. Then every periodic point in J is repelling or parabolic. Moreover, every cycle of Siegel disks for f contains at least one critical point on its boundary.

A remark on the above is that the proof in fact shows that at the critical point $c \in \partial\Delta$ in the boundary of the cycle of Siegel disks, at least two distinct rays land on it (at least in the case of a fixed Siegel disk, but extendable to cycles of Siegel disks).

Remark on proof of Lemma 18.12: Recall that as we are not assuming that the Julia set is locally connected (that is, that all rays land continuously onto J), we cannot affirm that the set $X \subset \mathbb{R}/\mathbb{Z}$ of angles whose rays land at z_0 is compact.

Despite this, since we are assuming that at least one periodic ray lands at z_0, we have that z_0 is periodic with period a divisor of the period of the ray. By first assuming that the ray is fixed, the set X is mapped to itself $n \mapsto nt$ in $\mathbb{R}/\mathbb{Z}$, injectively as z_0 is not a critical point, and surjectively because of lemma 18.1.

More generally, we have:

Proposition 18.1. *Suppose f is a polynomial map of degree n for which J is connected, and $z_0 \in J$ is not a critical point. Let X_{z_0} be the set of rays landing at z_0, corresponding to a set of angles in $\mathbb{R}/\mathbb{Z}$. Then the map $t \mapsto nt$ on $\mathbb{R}/\mathbb{Z}$ gives rise to a bijection $X_{z_0} \to X_{f(z_0)}$.*

Proof. For a sufficiently small neighborhood N of z_0, we have a conformal isomorphism $f|_N : N \to f(N)$, which maps a ray R_t to R_{nt}, and if R_t lands at z_0, then R_{nt} lands at $f(z_0)$. (Restatement of Lemma 18.1.) If t and t' are such that $nt \equiv nt' \mod \mathbb{Z}$, then the rays R_t and $R_{t'}$ map to the same ray in $f(N)$, contradicting injectiveness of $f|_N$. Hence $X_{z_0} \to X_{f(z_0)}$ is injective.

By considering the map $f|_N^{-1}$, we also have surjectiveness; if s is such that R_s lands at $f(z_0)$, then it is the image of some ray R_t for which $nt = s$. There are in principle n possible preimages of s under the map $t \mapsto nt$, but due to injectiveness, there must be a single one such t such that R_t lands at z_0. □

Corollary 18.2. *Under the previous hypotheses, but assuming now that z_0 is a critical point of local degree $m \geq 2$, the map $X_{z_0} \to X_{f(z_0)}$ is surjective and m to 1.*

How do we know the map must preserve the cyclic order? Is this just a consequence of the $t \mapsto nt$ map on the circle? Because the map may wind around the circle more than once. This is probably basic.

Note also that possibly $X_{z_0} = \emptyset$, and theorem 18.12 deals with the case of X_{z_0} containing a periodic ray. It follows that z_0 is periodic, X_{z_0} is finite and the rays all share the same period.

Remarks on the proof of Theorem 18.11 (Repelling and Parabolic Points are Landing Points):

Assume $f(0) = 0$ is repelling or parabolic, and let E be the set of all backwards orbits $\mathbf{z} = (z_0, z_1, z_2, \ldots)$, that is,

$$\cdots \mapsto z_2 \mapsto z_1 \mapsto z_1 \mapsto z_0$$

such that it converges non-trivially to 0 in backwards time. More precisely, the book claims that this means $\lim_{k\to\infty} z_k = 0$ and $z_k \neq 0$ for k large; but does this mean that *for all* k large, we have $z_k \neq 0$, or that for arbitrarily large k we will find some $z_k \neq 0$?

The former necessarily implies the latter, but they are in fact equivalent. Because we may take a sufficiently small neighborhood N of 0 such that, for all $k \geq k_0$, we always have $z_k \in N$, so that in the repelling case there is a unique preimage of z_k contained in N, due to the local conjugation to $w \mapsto \lambda w$, and the sequence will be non-zero for all bigger k; and in the parabolic case, the same happens, as the backwards sequence must fall inside a repelling petal and converge also non-trivially to 0. (This space E will also include all homoclinic orbits of 0 as well.)

According to the third edition, we may specify a topology for E. Consider neighborhoods V_0 and V_1 of 0 such that f maps $f|_{V_0} : V_0 \to V_1$ as a conformal isomorphism, with inverse $g : V_1 \to V_0$. (In the repelling case, we may stipulate that $\overline{V_0} \subset V_1$.)

Let

$$E_k \coloneqq \{\mathbf{z} = (z_0, z_1, \ldots) \mid z_j \in V_0 \cap V_1 \setminus \{0\},\ \forall j \geq k\},$$

where E_k is a subset of the set of all backwards orbits. Naturally

$$E_0 \subset E_1 \subset E_2 \subset \cdots,$$

and if $\mathbf{z} \in E_k$, then $\mathbf{z} \in E$. This is because by our possible choice of V_0 the sequence will fall within an attracting neighborhood of attracting petal for g, so that it will convergence non-trivially to 0. And in fact, if $\mathbf{z} \in E$, then for some k we must have that $\mathbf{z} \in E_k$ due to the convergence. This implies that E is the union of the ascending chain of sets E_k. (Can we imbue E with a direct limit topology?)

Now consider the projection $\pi_k : E_k \to V_0$, $\mathbf{z} \mapsto z_k$. This is an injection, given that we can recover the other coordinates of $\mathbf{z}$ from z_k; we have that $z_j = f^{k-j}(z_k)$ for $j \leq k$, and $z_j = g^{j-k}(z_k)$ for $j > k$, given that $z_j \in V_0 \cap V_1 \setminus \{0\}$ for all $j \geq k$. In fact, these other coordinates are holomorphic with respect to z_k. Moreover, the image of this projection is open in $\mathbb{C}$; if $\hat{z}$ is sufficiently close to z_k, so that in the repelling case $\hat{z}$ is still in the neighborhood $V_0 \cap V_1 \setminus \{0\}$ which is contracting for g, or in the same repelling petal as z_k, we may produce

$$(f^k(\hat{z}), f^{k-1}(\hat{z}), \ldots, f(\hat{z}), \hat{z}, g(\hat{z}), \ldots, g^j(\hat{z}), \ldots) \in E_k.$$

In the repelling case, $\pi_k : E_k \to V_0 \setminus \{0\}$ will be a bijection, and in the parabolic case, E_k will be mapped into the disjoint union of the repelling petals for 0 bijectively.

We then give each E_k a topology such that π_k is a homeomorphism into V_0, and give E the direct limit topology. Recall that $\psi : E \to Y$ is then continuous if and only if the restrictions $\psi|_{E_k} : E_k \to Y$ are continuous. Each E_k will also be a Riemann surface, and hence E will acquire a canonical conformal structure from the direct limit. Moreover, the maps $\pi_k : E \to \mathbb{C}$ will be all holomorphic, since for $\mathbf{z} \in E$, for some arbitrarily large $j > k$ we have $\mathbf{z} \in E_j$, and

$$\pi_k(\mathbf{z}) = f^{j-k}(\pi_j(\mathbf{z})),$$

where f^{j-k} and π_j are holomorphic on their domains.

We now consider the shift map

$$(z_0, z_1, z_2, \ldots) \mapsto (z_1, z_2, z_3, \ldots)$$

which maps E_k into E_{k-1} injectively, since we may recover z_0 from the other coordinates. It will also map E_0 into E_0. This means that $\sigma : E \to E$ is an injection. It has the inverse $\mathbf{f}$:

$$(z_0, z_1, z_2, \ldots) \mapsto (f(z_0), f(z_1), f(z_2), \ldots) = (f(z_0), z_0, z_1, \ldots)$$

such that

$$\mathbf{f}(\pi_k(\mathbf{z})) = \pi_k(\mathbf{f}(\mathbf{z})),$$

and will be a conformal isomorphism from E to E. Here $\mathbf{f}$ maps E_k to E_{k+1}.

In the repelling case, Königs linearization for g will give us that E is conformally isomorphic to $\mathbb{C} \setminus \{0\}$ through the map

$$\mathbf{k}(\mathbf{z}) = \lim_{k \to \infty} \lambda^k z_k.$$

Moreover, $\mathbf{f} : E \to E$ is conjugate to $z \mapsto \lambda z$ in $\mathbb{C} \setminus \{0\}$. Similarly, in the parabolic case, assuming $\lambda = 1$ by taking iterates of f, each component $E_{\mathcal{P}}$ corresponding to a repelling petal will be conformally isomorphic to $\mathbb{C}$ due to the Fatou coordinate $\alpha_{\mathcal{P}} : \mathcal{P} \to \mathbb{C}$, where $\mathbf{f}$ will be conjugate to $z \mapsto z + 1$.

If $\widetilde{K}$ is the set of $\mathbf{z} \in E$ such that all z_k belong to the filled Julia set, this will be equal to $\bigcap_{k \geq 0} \pi_k^{-1}(K) = \pi_0^{-1}(K)$, which is therefore closed in E. Moreover, $\widetilde{K}$ will be totally invariant in E under $\mathbf{f}$.

The philosophy behind both of these constructions seems to be that we can "blow up" the small scale repelling dynamics near the fixed point to a global dynamics on $\hat{\mathbb{C}}$, and have it be conjugate to our standard models. With this, we also blow up the "local Julia set" at the repelling fixed point to this global dynamics. In fact, it seems these constructions can be reproduced for any repelling or parabolic fixed point for all $f : S \to S$ holomorphic.

What do the homoclinic orbits look like in E? We can consider the preimage $\pi_0^{-1}(0)$ as a closed set, or the union of $\pi_k^{-1}(0)$ for all k. In fact

$$\pi_0^{-1}(0) \supset \pi_1^{-1}(0) \supset \pi_2^{-1}(0) \supset \cdots \supset \pi_k^{-1}(0) \supset \cdots$$

Another small detail in the proof of Lemma 18.14: how do we know that the map $\pi_0 : U_0 \to \mathbb{C} \setminus K$ is surjective? Given $\hat{z} \in \mathbb{C} \setminus K$, we know that the set of preimages of $\hat{z}$ accumulates at $J = \partial \mathcal{A}(\infty)$, so that some iterated preimage falls within the described neighborhoods of the repelling fixed point $0 \in J$. This shows that $\pi_0 : E \setminus \widetilde{K} \to \mathbb{C} \setminus K$ is surjective and conformal, but why is it surjective on each component? The proof in the book (specially second edition) shows that $\pi_0 : E \setminus \widetilde{K} \to \mathbb{C} \setminus K$ is a holomorphic covering map. If U_0 is a component of $E \setminus \widetilde{K}$, then $\pi_0|_{U_0} : E \setminus \widetilde{K} \to \mathbb{C} \setminus K$ will also be a covering map. Otherwise, if $\hat{z} \in \partial \pi_0(U_0)$ and N is a sufficiently small evenly covered neighborhood of $\hat{z}$, let $\widetilde{N}_i$ be the lifts of N. N will contain points in $\pi_0(U_0)$, so by taking $\mathbf{z} \in U_0$ such that $\pi_0(\mathbf{z}) \in N$, we consider the lift $\widetilde{N}_i$ that contains $\mathbf{z}$. But since N also contains points not in $\pi_0(U_0)$, $\widetilde{N}_i$ will contain points in $(E \setminus \widetilde{K}) \setminus U_0$. But as $\widetilde{N}_i$ is connected, it must have been fully contained in U_0, a contradiction.

Recall that each component U of $E \setminus \widetilde{K}$ is invariant under $\mathbf{f}$. In the repelling case, by the conformal isomorphism $E \cong \mathbb{C} \setminus \{0\}$ where $\mathbf{f}$ is conjugate to $z \mapsto \lambda z$, for every punctured disk $\mathbb{D}_r^* \subset \mathbb{C} \setminus \{0\}$, every component of $E \setminus \widetilde{K}$ will intersect $\mathbb{D}_r^*$.

19 Hyperbolic and Subhyperbolic Maps

Remark on the proof of 19.1: Given the neighborhood V' of J which is conformally hyperbolic and on which the expanding factor of f is $\geq k$, it is claimed that there exists an $\varepsilon > 0$ such that both $f^{-1}(N_\varepsilon(J)) \subset V'$, and for each $z \in N_\varepsilon(J)$, there exists at least one minimal μ-geodesic in V' joining z to J.

The first claim follows from from compactness of J and that inverse branches of f are defined on a neighborhood of J. Since J must contain no critical points, it also cannot contain any critical values. Hence for all

$w \in J$ there exists $\varepsilon_w > 0$ such that there exists d inverse branches of f^{-1} defined on $N_{\varepsilon_w}(w)$, giving rise to conformal isomorphisms onto neighborhoods of the preimages of w; by continuity of these inverse branches, we may assume that each of these images is contained in V', possibly decreasing ε_w. By compactness, we may take an uniform $\varepsilon > 0$.

For the second claim, each point $w \in J$ will have a well defined injectivity radius $r_w > 0$ such that the exponential map is a diffeomorphism from a neighborhood $B_{r_w}(0_w)$ of $0_w \in T_w V'$ onto a normal neighborhood of w, and by compactness, we may take an uniform radius $r > 0$. We may also take totally normal convex neighborhoods, where distances between points are realized by geodesics within the neighborhood.

Moreover, as a consequence of the expanding behavior on the neighborhood, we in fact have that

$$f^{-1}(N_\varepsilon(J)) \subset N_\varepsilon(J),$$

as claimed afterwards, given that for $z \in f^{-1}(N_\varepsilon(J))$,

$$d(z, J) \leq d(f(z), J)/k < \varepsilon/k.$$

Later in the proof, it is claimed that if $\hat{z}$ is an accumulation point for an orbit in the Fatou set must be in the Fatou set, and it cannot be a rotation domain, following Theorem 11.17 and Lemma 15.7. Theorem 11.17 states that the boundary of any Siegel disk or cycle of Siegel disks is contained in the closure $\overline{P}$ of the postcritical set. If f had a rotation domain in the first place, by the above, the critical point accumulating on its boundary would have to be contained in the Julia set, which is excluded by the dynamically hyperbolic neighborhood V' of J.

What is the radial derivative?

$$\frac{d \log F(w)}{d \log w} = w \frac{F'(w)}{F(w)}$$

Maybe one way to interpret this is via the following. If $w = re^{i\theta}$, so that $z = \log w = \log r + i\theta$, we may consider

$$\frac{d \log f(e^z)}{dz} = \frac{1}{f(e^z)} f'(e^z) e^z,$$

being just the complex derivative of $g(z) = \log f(e^z)$. But this will coincide with the first partial derivative $\partial_x g$ of g; this is varying the real part of z, that is, the radial part of w (up to a logarithm). The importance of $g(z) = \log f(e^z)$ is that (locally) it is a lifting of f via the exponential map.

Remark on the proof of Theorem 19.2: Connectedness of the Julia set plays the following role in the proof: As $J = X \subseteq S^2$ is a compact connected subset, every component of its complement is open and simply connected. This means that we can indeed apply Lemma 19.3.

Problem (19-a. The non-wandering set)**.** By definition, the *non-wandering set* for a continuous map $f : X \to X$ is the closed subset $\Omega \subset X$ consisting of all $x \in X$ such that for every neighborhood U of x there exists an integer $k > 0$ such that $U \cap f^k(U) \neq \emptyset$. Show that the non-wandering set for a rational f is the disjoint union of its Julia set, its rotation domains (if any), and its set of attracting periodic points.

Proof. The non-wandering set for a general continuous map is closed from the definition, since the complement of the non-wandering set has an "open" definition. Naturally the Julia set J is contained in the non-wandering set, since the set of all iterated preimages of a point $z \in J$ is dense; and also the set of all attracting periodic orbits. If a point z belongs to a rotation domain, since irrational rotations are dense in the circle, this implies that z is non-wandering.

Now suppose z is in the Fatou set, but is neither in a rotation domain nor it is an attracting periodic point. If the Fatou component U to which z belongs is not periodic, then z is wandering. If U is periodic, then it either is the immedaite basin of an attracting cycle, or of a parabolic cycle. It is sufficient to consider the case of a fixed point. Since small neighborhoods of z within U converge uniformly to either the attracting fixed point of or to the parabolic fixed point, z is wandering. □

Problem (19-b. Axiom A)**.** In the literature on smooth dynamical system a 1-dimensional map is said to satisfy Smale's Axiom A if and only if the following two conditions are satisfied:

(1) The non-wandering set Ω splits as the union of a closed subset Ω^+ on which f is infinitesimally expanding with respect to a suitable Riemannian metric, and a closed subset Ω^- on which f is contracting.

(2) Periodic points are everywhere dense in Ω.

Show that a rational map is hyperbolic if and only if it satisfies Axiom A.

Proof. If f is hyperbolic, then it cannot admit rotation domains, so Ω is the disjoint union of the Julia set with the attracting periodic cycles. Naturally f is expanding on a neighborhood of J, contracting on the attracting cycles, and periodic points are everywhere dense in Ω, so that f satisfies Axiom A.

Conversely, if f satisfies Axiom A, it cannot have rotation domains by the requirement of density of periodic points. Moreover, the splitting of Ω must be with respect to the Julia set J and the attracting cycles, giving rise to hyperbolicity on a neighborhood of J. $\square$

Problem (19-d. An orbifold example)**.** (1) Show that the Julia set for the rational map $z \mapsto (1 - 2/z)^{2n}$ is the entire Riemann sphere.

(2) For $n > 1$, show that the orbifold metric in this example is hyperbolic.

(3) For $n = 1$, show it is euclidean.

Proof. We compute the critical points of $f(z) = (1 - 2/z)^{2n}$ and its local degrees. Note that

$$f'(z) = 2n\left(1 - \frac{2}{z}\right)^{2n-1}\frac{2}{z^2},$$

$\square$

so that $z = 2$ is a finite critical point for f, where $f(2) = 0$. Moreover,

$$\frac{f'(z)}{f(z)} = \frac{4n}{z(z-2)},$$

and

$$\operatorname{ord}(f, 2) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz = \frac{4n}{2\pi i}\int_\gamma \frac{1}{2(z-2)} - \frac{1}{2z}dz = \frac{2n}{2\pi i}\int_\gamma \frac{1}{z-2}dz = 2n,$$

where γ is a small loop around 2. Thefore the local degree of f at the critical point $z = 2$ is $2n - 1$. Note that the degree of f is $2n$, so that it has $4n - 2$ critical points counted with multiplicity.

In order to take account of the other critical points, note that $f^{-1}(\infty) = \{0\}$ and $f(\infty) = 1$. This means that 0 is the other critical point, with multiplicity $2n - 1$, and these are all critical points. Since we have the preperiodic orbit

$$2 \mapsto 0 \mapsto \infty \mapsto 1 \mapsto 1 \mapsto 1 \mapsto \cdots,$$

and 1 is a repelling fixed point since

$$f'(1) = 2n(-1)^{2n-1}\frac{2}{1^2} = -4n.$$

Due to the classification of Fatou components and their relationship with the postcritical set, f can have no rotation domains, no parabolic basins, no Cremer points and no attracting periodic orbits. Hence $J = \hat{\mathbb{C}}$.

In order to compute the orbifold associated to f, we see that $S = \hat{\mathbb{C}}$, since f has no attracting periodic orbits, and we recall that the following condition for (S, ν) must be satisfied:

For all $z \in S$, $\nu(f(z))$ must be a multiple of of $n(f, z)\nu(z)$.

Also, the ramification indexes are not 1 only at the postcritical points. The minimal ν that satisfies this condition is then

$$\nu(2) = 1, \quad \nu(0) = 2n, \quad \nu(\infty) = 4n^2, \quad \nu(1) = 4n^2.$$

Hence

$$\chi(S, \nu) = \chi(\hat{\mathbb{C}}) + \frac{1}{2n} + \frac{1}{4n^2} + \frac{1}{4n^2} - 3 = \frac{1}{2n} + \frac{1}{2n^2} - 1.$$

For $n = 1$, we see that the orbifold is euclidean, and for $n > 1$ the orbifold is hyperbolic, following as in the proof of 19.6. f will be expanding with respect to the hyperbolic orbifold metric on the entire sphere.

As in the proof of 19.9, if $n = 1$ with euclidean orbifold, we have that $f : (S, \nu) \to (S, \nu)$ is a 2-fold orbifold covering,

Problem (19-e. Expanding maps)**.** A map from a metric space to itself is said to be *expansive* on a subset X if there exists $\varepsilon > 0$ so that, for any two points $x \neq y$ whose orbits remain in X forever, there exists some $k \geq 0$ so that $f^k(x)$ and $f^k(y)$ have distance greater than ε. Using Sullivan's results, show that a rational map is expansive on some neighborhood of its Julia set if and only if it is hyperbolic. (However, a map with a parabolic fixed point may be expansive on the Julia set itself.)

Proof. Suppose that f is hyperbolic, and V' and $N_\varepsilon(J)$ are sets as in the proof of Theorem 19.1. Suppose x and y are points in $N_\varepsilon(J)$ whose orbits stay in $N_\varepsilon(J)$ forever; necessarily then $x, y \in J$. If we take $\varepsilon > 0$ so small that if $d(z, w) < 2\varepsilon$ implies that there exists at least one minimal geodesic connecting z and w for arbitrary $z, w \in N_\varepsilon(J)$, then if we assume by contradiction that $d(f^n(x), f^n(y)) \leq \varepsilon$ for all n, we see that

$$d(x, y) \leq d(f^n(x), f^n(y))/k^n < \varepsilon/k^n,$$

where $k > 1$ is the expanding factor on V', by pulling back the geodesic connecting $f^n(x)$ and $f^n(y)$ by f^n. But this would imply that $d(x, y) \to 0$, so that $x = y$, a contradiction.

Now assume that f is expansive on some neighborhood V of the Julia set. We see that f cannot admit rotation domains. Otherwise, if U is a rotation domain and $\phi : U \to \mathbb{D}$ or A_r is a conformal isomorphism to the open disk or an open annulus conjugating f to an irrational rotation, this isomorphism will be a uniformly continuous homeomorphism on compact sets of U, and in particular on the foliated orbit closures of f in U. Hence the distortion of distances is bounded above and below in a controlled manner.

More precisely, suppose $x, y \in U \cap V$ belong to the same orbit leaf, so that $d(\phi(x), \phi(y)) < \delta$ in the usual euclidean distance, and $d(\phi(f^n(x)), \phi(f^n(y))) < \delta$. Then $d(f^n(x), f^n(y)) < \varepsilon'$ for all n, contradicting expansiveness, for a sufficiently small δ.

If $x \in V \setminus J$, then a small neighborhood of x (or more precisely, a subsequence of iterates) converges uniformly to an attracting or parabolic cycle. Suppose it converges to an attracting cycle (we may consider the case of a fixed point p). If x is to have its orbit contained in V, then $p \in \overline{V}$. But for a sufficiently big k, the two points $f^k(x)$ and $f^{k+1}(x)$, forming a Cauchy sequence, will contradict the hypothesis of expansiveness.

Similarly, if p is a parabolic fixed point, since $p \in J \subset V$, we similarly have a contradiction with expansiveness by considering a small enough attracting petal for p. This implies that if $x \in V$ is to have its orbit contained in V, then $x \in J$. This also implies that for f to be expansive on a neighborhood of J, there can also be no parabolic cycles.

Using Theorem 19.1, in order to show that f must be hyperbolic, we only need to show that J cannot contain a critical point. But having a critical point would contradict expansiveness on a neighborhood of it in J (how so?). □

Problem (19-f. Locally connected sets in the 2-sphere)**.** Give a complete characterization of compact locally connected subsets of the 2-sphere as follows.

(1) Prove the following theorem of Torhorst: If $X \subset S^2$ is compact and locally connected, then the boundary of every complementary component must be locally connected.

(2) Furthermore, prove that: If there are infinitely many complementary components, then their diameters tend to zero.

(3) Now using Lemma 19.5, conclude that these two conditions are necessary and sufficient for local connectivity.

Proof. (1) If X is locally connected, then it must have at most finitely many connected components. Otherwise we could find a sequence of points $x_n \in X$ belonging to distinct components accumulating at $x \in X$. This contradicts local connectedness at x, since any small neighborhood of x in X is disconnected.

We may therefore separate these components of X by a positive distance, and by cutting out neighborhoods of these components in the sphere, reduce to the case of X connected. This also implies that any connected component U of $S^2 \setminus X$ is open and simply connected, and each boundary ∂U is connected.

We therefore have a conformal isomorphism $\phi : \mathbb{D} \to U$, which extends to the boundary $\phi : \overline{\mathbb{D}} \to \overline{U}$ if and only if ∂U is locally connected, if and only if $S^2 \setminus U$ is locally connected.

Let $x \in \partial U$, and assume that ∂U is not locally connected at x. That is, all sufficiently small neighborhoods of x (within ∂U) are disconnected.

(2) Let $(x_n)_n$ be a sequence of points such that $x_n \in \partial U_n$, where the U_n are distinct components of $S^2 \setminus X$. By taking subsequences, we may assume that $x_n \to x$. As the connected components of $S^2 \setminus X$ are open, we must have that $x \in X$.

As X is locally connected at x, for every $\varepsilon > 0$, we may find an open neighborhood N of x in S^2 so that $N \cap X$ is connected and $\operatorname{diam}(N \cap X) < \varepsilon$. We know that for all $n \geq n_0$, we have $x_n \in N$, so that $\partial U_n \cap N \neq \emptyset$, and consequently $U_n \cap N \neq \emptyset$.

(incomplete; have not had much progress. The picture of item (2) should be clear, and the proof showuld follow from local connectedness at the points.)

(One possible idea of proof for item 1 is to show that ∂U must be a retract of X; must this seems too strong.) □

Appendix A: Theorems from Classical Analysis

I am slightly confused as to how exactly we are taking the branches of the square root in the proof of Lemma A.6. We have a conformal isomorphism $\psi : \mathbb{D} \to U$ mapping $0 \mapsto 0$ and $\psi'(0) = 1$ in the η-plane (ψ is *schlicht*). We also have the map $g(w) = 1/w^2$, mapping $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto $\mathbb{D}$ in a two to one covering branched over ∞.

It may be easier at first to consider just disks and forget about infinity. Let $f : \mathbb{D} \to \mathbb{D}$ be the double branched cover $f(w) = w^2$, so that we have the composition $\psi \circ f : \mathbb{D} \to U$ as a double branched cover. The square root map $f(z) = z^2$ actualy double covers the whole of $\mathbb{C}$, so that we may consider the branched cover $f|_{f^{-1}(U)} : f^{-1}(U) \to U$, which contains a neighborhood of U. we then have the diagram

$$\begin{array}{ccc} \mathbb{D} & & f^{-1}(U) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{D} & \xrightarrow{\psi} & U \end{array}$$

where the map $\psi \circ f : \mathbb{D} \to U$ will lift to an isomorphism $\widetilde{\psi} : \mathbb{D} \to f^{-1}(U)$. (This can maybe be see first for the unbranched covers $\mathbb{D} \setminus \{0\} \to \mathbb{D} \setminus \{0\}$ and $f^{-1}(U) \setminus \{0\} \to U \setminus \{0\}$, and then filling in the punctures at 0.) By the coordinate change $1/z$, we obtain the desired maps on a neighborhood of infinity on the double covers.

Note that the argument above can be readily generalized to $f_n(z) = z^n$ for $n \geq 1$. Does this gives us more inequalities?

Problem (A-1. Area of the filled Julia set)**.** Consider the polynomial map $f_c(z) = z^2 + c$. Let $w = \hat{\phi}(z)$ be the associated Böttcher map mear infinity,and let $z = \psi(w)$ be the inverse map.

(1) In analogy with the equation (9:5), show that ψ satisfies the identity

$$\psi(w^2) = \psi(w)^2 + c,$$

and conclude that ψ has Laurent series of the form

$$\psi(w) = w\left(1 + \frac{p_1(c)}{w^2} + \frac{p_2(c)}{w^4} + \frac{p_3(c)}{w^6} + \cdots\right)$$

where each $p_k(c)$ is a polynomial of degree k with rational coefficients.

(2) Let K_c be the filled Julia set for f_c. Show that the area of K_c is upper semicontinuous as a function of c.

(3) If K_c is connected, or in other words if c belongs to the Mandelbrot set, show by Lemma A.4 that its area is given by the formula

$$A(K_c) = \pi(1 - |p_1(c)|^2 - 3|p_2(c)|^2 - 5|p_3(c)|^2 - \cdots).$$

(4) On the other hand, show that the previous equation breaks down when K_c is not connected. In fact, the left side is zero but the right side is $-\infty$. Show that the sum

$$|p_1(c)|^2 + |p_2(c)|^2 + 5|p_3(c)|^3 + \cdots$$

is infinite. In fact, when K_c is not connected, show that ψ cannot be extended as a holomorphic function over all of $\mathbb{C} \setminus K_c$, and conclude that the sequence of coefficients $p_1(c), p_2(c), \dots$ must be unbounded. The area is zero in this case, since K_c coincides with the Julia set and since f_c is hyperbolic.

Proof. Recall that on a neighborhood of a superattracting fixed point, the Böttcher map conjugates the function to the n-th power map $w \mapsto w^n$. This conjugation will be preserved by considering the superattracting fixed point at ∞ for a polynomial, conjugating it to the map $w \mapsto w^d$ where d is the degree, outside a large disk. As f_c is monic, we have that ψ, mapping a neighborhood of infinity on the $\mathbb{C} \setminus \overline{\mathbb{D}}$ plane to a neighborhood of infinity in the $\mathcal{A}(\infty) = \mathbb{C} \setminus K_c$ plane, will be of the form

$$\psi(w) = w + b_0 + \frac{b_{-1}}{w} + \frac{b_{-2}}{w^2} + \cdots$$

so that $\psi(w) \sim w$ as $w \to \infty$, and satisfies

$$\psi(w^2) = \psi(w)^2 + c.$$

Also recall that the local inverse to the Böttcher map ψ extends to a maximal disk $\mathbb{C} \setminus \overline{\mathbb{D}}_r$, $r \geq 1$, and $r = 1$ if and only if $\hat{\phi}$ extends to a conformal isomorphism $\hat{\phi} : \mathbb{C} \setminus K \to \mathbb{C} \setminus \overline{\mathbb{D}}$, if and only if $\mathcal{A}(\infty)$ contains no critical points, if and only if K_c is connected.

We have that

$$\psi(w^2) = w^2 + b_0 + \frac{b_{-1}}{w^2} + \frac{b_{-2}}{w^4} + \frac{b_{-3}}{w^6} + \cdots$$

and

$$c + \psi(w)^2 = c + \left(\sum_{n \leq 1} b_n w^n \right)^2 = c + \sum_{m \leq 2} \left(\sum_{k+l=m} b_k b_l \right) w^m$$

which becomes

$$\psi(w)^2 + c = w^2 + 2b_0 w + (b_0^2 + 2b_{-1} + c) + (2b_{-2} + 2b_0 b_{-1}) \frac{1}{w} + \cdots .$$

We conclude that $b_0 = 0$, $b_{-1} = -c/2$, and $b_2 = 0$. In general, we have the recursive relations

$$b_m = \sum_{k+l=2m} b_k b_l = b_1 b_{2m-1} + b_0 b_{2m} + b_{-1} b_{2m+1} + \cdots + b_{2m-1} b_1$$

for $m \leq 0$, except for $m = 0$, and

$$0 = \sum_{k+l=2m+1} b_k b_l = b_1 b_{2m} + b_0 b_{2m+1} + b_{-1} b_{2m+2} + \cdots + b_{2m} b_1.$$

for $m \leq 0$. (Note that the solutions $k+l = 2m$ and $l+k = 2m$ are distinct in the sum for k and l distinct.) The second set of relations imply, by induction, that for m even, $b_m = 0$. From the first set of relations, and given that $b_1 = 1$, $b_{-1} = -c/2$ and $b_{2m} = 0$, we have that

$$b_m = 2b_1 b_{2m-1} + 2b_0 b_{2m} + 2b_{-1} b_{2m+1} + \cdots + 2b_{m+1} b_{m-1} + b_m^2.$$

This gives us

$$b_{2m-1} = \frac{b_m - b_m^2}{2} - (b_0 b_{2m} + b_{-1} b_{2m+1} + \cdots + b_{m+1} b_{m-1}).$$

As $p_k(c) = b_{-2k+1}$, we get that $b_{-1} = p_1(c) = -c/2$, and

$$b_{-3} = p_2(c) = \frac{b_{-1} - b_{-1}^2}{2} - b_0 b_{-2} = \frac{c^2}{8} - \frac{c}{4},$$

$$b_{-5} = p_3(c) = \frac{b_{-2} - b_{-2}^2}{2} - (b_0 b_{-4} + b_{-1} b_{-3}) = \frac{c^3}{16} - \frac{c^2}{8}.$$

It is straightforward to conclude by induction that $p_k(c)$ is indeed a rational polynomail of degree k with rational coefficients.

Recall that a function $f : X \to \mathbb{R}$ is upper-semicontinuous at a point x_0 if $\limsup_{x \to x_0} f(x) \leq f(x_0)$. Equivalently, for all $y > f(x_0)$, there is a neighborhood U of x_0 such that $f(x) < y$ for all $x \in U$. Moreover, f is upper semicontinuous if for all $y \in \mathbb{R}$, the set $f^{-1}(-\infty, y)$ is open.

If $c \in \mathbb{C} \setminus M$, then $A(c) = 0$ (why? hyperbolicity?), and this is an open set. If $c \in M$, then K_c is connected, and we have the conformal isomorphism $\psi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$. Lemma A.4 implies that the area $A(c) = A(K_c)$ is given by

$$\pi - \pi \sum_{k=1}^{\infty} (2k-1) |p_k(c)|^2.$$

Suppose that $g_k : K \to [0, +\infty)$ is a sequence of non-negative continuous functions on a compact Hausdorff set K, and let $g(x) := \sup_{k \geq 0} g_k(x)$. We show that g is lower semicontinuous. For $y \in \mathbb{R}$, note that $g(x) > y$ if and only if there exists some k such that $g_k(x) > y$, which means that

$$g^{-1}(y, +\infty) = \bigcup_{k \in \mathbb{N}} g_k^{-1}(y, +\infty).$$

More generally, the supremum of a sequence of lower semicontinuous functions is lower semicontinuous, and the infimum of a sequence of upper semicontinuous functions is upper semicontinuous.

As $\pi - A(c)$ is given by a series of non-negative terms, it can be realized as a supremum of the partial sums of the polynomials, so that it will be lower semicontinuous. This concludes that $A(c)$ is upper semicontinuous for $c \in M$, and this glues with the result for $c \notin M$.

If $c \in M$, so that K_c is connected, by the lemma A.4 we immediately get that

$$A(K_c) = \pi(1 - |p_1(c)|^2 - 3|p_2(c)|^2 - 5|p_2(c)|^2 - \cdots).$$

On the other hand, if K_c is not connected, we know that $A(K_c) = 0$ (by hyperbolicity?), and that the local inverse to the Böttcher map ψ has a maximal extension to $\mathbb{C} \setminus \overline{\mathbb{D}_r}$, where $r > 1$. This means that the Laurent series

$$\psi(w) = w\left(1 + \frac{p_1(c)}{w^2} + \frac{p_2(c)}{w^4} + \frac{p_3(c)}{w^6} + \cdots\right)$$

diverges for $|w| < r$. Letting $\zeta = 1/w$, we have that the power series

$$1 + p_1(c)\zeta^2 + p_2(c)\zeta^4 + p_3(c)\zeta^6 + \cdots$$

has radius of convergence $R = 1/r$. Hence

$$R = \frac{1}{\limsup \sqrt[2n]{|p_n(c)|}} \implies r = \limsup \sqrt[2n]{|p_n(c)|},$$

so that for all $\varepsilon > 0$, there exists infinitely many n for which

$$\sqrt[2n]{|p_n(c)|} \geq r - \varepsilon \implies |p_n(c)| \geq (r - \varepsilon)^{2n},$$

and by taking $\varepsilon < r - 1$, we get that the sequence of coefficients $p_n(c)$ is unbounded. This concludes that the formula on the right hand side of the area $A(K_c)$ diverges to $-\infty$ for K_c not connected. □

Appendix B: Length-Area Modulus Inequalities

We recall the the definition of the module of a quadrilateral. If Ω is a Jordan domain with $z_1, z_2, z_3, z_4 \in \partial\Omega$ in cyclic order, then $Q = Q(z_1, z_2, z_3, z_4)$ is a quadrilateral. One possible way to define its module is as follows. Let Γ be the family of (locally rectifiable) arcs joining the sides (z_1, z_2) and (z_3, z_4)

(the "a-sides"), and let P be the family of conformal metrics $\rho : U \to \mathbb{R}_{\geq 0}$ whose area $A(\rho) := \iint_Q \rho^2 dxdy$ is $\neq 0, \infty$. If $L_\rho(\gamma) = \int_\gamma \rho(z)|dz|$, then

$$M(Q) = \frac{a}{b} = \inf_{\rho \in P} \frac{A_\rho(Q)}{(\inf_{\gamma\in\Gamma} L_\rho(\gamma))^2},$$

where Q is conformally isomorphic to a rectangle $[0,a]\times[0,b]$ mapping the a-sides to a-sides, and b-sides to b-sides. This is proved via the same length-area estimates (using Cauchy-Schwarz) as in the appendix.

Now, if Γ is any family of (locally rectifiable) paths in a Riemann surface S, and P is the family of (locally integrable) conformal metrics on S whose area is $A_\rho(S) \neq 0, \infty$ (where locally $\rho = \rho(z)|dz|$, we define the extremal length of Γ as

$$\mathcal{L}(\Gamma) = \sup_{\rho\in P} \frac{(\inf_{\gamma\in\Gamma} L_\rho(\gamma))^2}{A_\rho(S)} = \sup_{\rho\in P} \frac{\left(\inf_{\gamma\in\Gamma} \int_\gamma \rho(z)|dz|\right)^2}{\iint_S \rho^2}.$$

Note that the module of a quadrilateral is the inverse of the extremal length of the paths connecting the a-sides, but is equal to the extremal length of the paths connecting the b-sides.

Most of the results of the chapter consists of finding curves satisfying the inequality with respect to the modulus of a quadrilateral of extremal length of curves, specially in comparison with the euclidean metric.

Problem (B-4. Branner-Hubbard criterion)**.** Let $K_1 \supset K_2 \supset K_3 \supset \cdots$ be compact conneted subsets of $\mathbb{C}$ with each K_{n+1} contained in the interior of K_n. Suppose further that the interior $\operatorname{int} K_n$ is simply connected, so that each difference $A_n \operatorname{int} K_n \setminus K_{n+1}$ is an annulus.

(1) If $\sum_{n=1}^{\infty}$ mod (A_n) is infinite, show that the intersection $\bigcap K_n$ reduces to a single point.

(2) Show that the converse statement is false: this intersection may reduce to a single point even though $\sum_{n=1}^{\infty}$ mod $(A_n) < \infty$. (As a first step, you would consider the open unit disk $\mathbb{D}$ and a closed disk $\overline{\mathbb{D}'}$ of radius $0 < r < 1$ centered at $1 - r - \varepsilon$, showing that mod $(\mathbb{D} \setminus \overline{\mathbb{D}'})$ tends to 0 as $\varepsilon \to 0$.)

Proof. We have previously proved that the intersection $K = \bigcap K_n$ is nonempty, compact and connected. If we only assume that the interiors $\operatorname{int} K_n$ are simply conneted, this does not give us that $\operatorname{int} K$ is simply connected; consider for example the filled Julia set for $f(z) = z^2 - 1$, whose interior has countably many components. Can we prove that K is simply connected?

Consider the annulus $B_n = \bigcup_{i=1}^{n} A_i = \operatorname{int} K_1 \setminus K_{n+1}$, so that the B_n form an ascending chain of annuli such that $\mod(B_n) \to \infty$. By Corollary B.8, we have

$$4\operatorname{diam}(K_{n+1})^2 \leq \frac{\operatorname{Area}(\operatorname{int} K_1)}{\mod(B_n)} \to 0.$$

This implies that K must have diameter 0, and consists of a single point.

(Honestly, I don't want to prove the second part right now.) □

Can we prove that K is simply connected? Path connected? Locally connected? Or that $B = \bigcup_{i=1}^{\infty} A_i$ is a topological annulus? I think B being a topological annulus is by virtue of K being connected.

Naturally K is not necessarily locally connected, otherwise MLC would have already been solved. I am also willing to assume that K is not necessarily path connected.

Orsay Notes: If K is locally connected, then it will be arc-connected.

Appendix E: Branched Coverings and Orbifolds

Recall that for a general non-constant holomorphic map $f : S \to T$, for $\hat{z} \in S$, we may find charts around $\hat{z}$ and $f(\hat{z})$ such that f is locally of the form $z \mapsto z^d$, where d is the local degree of f at $\hat{z}$ and does not depend on the choice of charts. $\hat{z}$ is a critical point/branch point and $f(\hat{z})$ is a critical value/branch value if $d > 1$. Hence f is a local homeomorphism outside of the critical points. Naturally the set of critical points is closed and discrete; however, it does not imply that the set of critical values is closed and discrete. This will be the case, however, if f is proper.

We already know several properties of non-constant proper holomorphic maps, most notably that they have a well defined branching number, that is, the number of elements in a generic fiber, so that the map will be an n-sheeted holomorphic covering map (Forster).

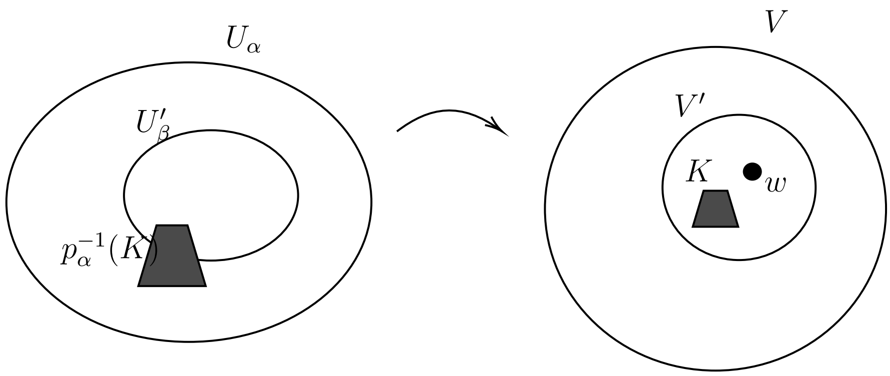
A holomorphic map $p : S \to T$ is a *branched covering map* (BCM), in Milnor's definition, if every point w of T has a connected neighborhood W such that, for each component U of $p^{-1}(W)$, the map $p|_U : U \to W$ is proper (and in particular surjective). This is more general than requiring that p be a (holomorphic) covering map, where we require that $p|_U : U \to W$ be a conformal isomorphism, and more general than a usual proper holomorphic map.

For usual covering maps, all points in the domain must have local degree equal to 1, whereas in a branched covering, the degree can be ≥ 1. A standard example is $z \mapsto z^d$ in $\mathbb{D}$.

Branched covering maps will be surjective, but may not themselves be proper; one example is the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$, where it is infinity to one.

If $w \in S$ and W is a connected neighborhood of w such that the property is satisfied (we will call W a BCM-neighborhood of w), is it the case that for another open connected set $W' \subset W$ containing w we have that W' satisfies the property?

Proposition 19.1. *Let $p : S \to T$ be a BCM and, for $w \in T$, W be a BCM-neighborhood of w. Then for all open connected neighborhoods $W' \subset W$ of w, W' is a BCM-neighborhood of w.*



Proof. If $p^{-1}(W) = \bigsqcup U_\alpha$ and $p^{-1}(W') = \bigsqcup U'_\beta$, for all β there is a unique α such that $U'_\beta \subseteq U_\alpha$. In order to show that $p|_{U'_\beta} : U'_\beta \to W'$ is proper, let $K \subseteq W'$ be compact. We prove that $p_\alpha^{-1}(K) \cap U'_\beta$ is compact. Consider A an open neighborhood of K such that $\overline{A} \subset W'$, which always exists since T is a normal Hausdorff space, and let $(z_n)_n$ be a sequence in $p_\alpha^{-1}(K) \cap U'_\beta$. Since $p_\alpha^{-1}(K)$ is compact, there exists a subsequence $z_{n_k} \to z \in p_\alpha^{-1}(K)$. Then $p(z_{n_k}) \to p(z) \in K$. If $z \notin U'_\beta$, then $z \in \partial U'_\beta$. We show that $p(z) \in \partial W'$. Since $p(z_{n_k}) \in W'$, we have $p(z) \in \overline{W'}$. If it were the case that $p(z) \in W'$, then the set $\{p(z), p(z_{n_k})\}$ would be compact in W', and its pre-image would be a compact set in U'_β. But evidently (z_{n_k}) would be a sequence that has no accumulation point in U'_β. So we have that $p(z) \in \partial W'$. (This is part of a more general argument that for proper maps, boundaries map to boundaries in a specific sense.)

However, $\overline{A}$ is disjoint from $\partial W'$, and $p(z) \in \overline{A}$. This is a contradiction, hence we have that $z \in U'_\beta$, and the map $p_\beta : U'_\beta \to W'$ is proper. $\square$

Since, for general non-constant holomorphic maps, the set of critical values/ramification points is closed and discrete, given a BCM $p : S \to T$ and a point $w \in T$, we may take a BCM-neighborhood W of w on which w is possibly the only ramification point of p. We may also take W to be conformally isomorphic to the unit disk $\mathbb{D}$.

If U_α are the connected components of $p^{-1}(W)$, $p|_{U_\alpha} : U_\alpha \to W$ is proper, holomorphic and non-constant, and it is a $d_{w,\alpha}$-sheeted covering of W, possibly ramified only at w, where $d_{w,\alpha}$ is a positive integer depending on the point w and the component U_α of $p^{-1}(W)$. Then $p_\alpha : U_\alpha \setminus p^{-1}(w) \to W \setminus \{w\} \cong \mathbb{D}^*$ is a proper covering map.

But we know (Forster, 5.11) that p_α is a $d_{w,\alpha}$-sheeted covering of the unit disk that is conformally conjugate to $z \mapsto z^{d_{w,\alpha}}$, and by Riemann's removable singularity theorem, we have that $p^{-1}(w)$ consists of a single point $z_\alpha \in U_\alpha$ whose local degree $n(z_\alpha)$ is $d_{w,\alpha}$. It is worth noting that this conjugation is not simultaneous for all U_α above W. To summarize:

Proposition 19.2. *Let $p : S \to T$ be a branched covering map. If $w \in T$ and W is an open connected neighborhood of w such that:*

1. *If $\bigsqcup_\alpha U_\alpha$ are the connected components of $p^{-1}(W)$, then $p|_{U_\alpha} : U_\alpha \to W$ is a proper non-constant holomorphic map;*
2. *W is conformally isomorphic to the unit disk $\mathbb{D}$;*
3. *W is small enough such that w is possibly the only ramification point in W;*

Then, for each component U_α, there is a single element $z_\alpha \in U_\alpha$ on the fiber $p^{-1}(w)$, and $p|_{U_\alpha} \to W$ is conformally conjugate to the map $\mathbb{D} \to \mathbb{D}$ given by $z \mapsto z^{d_{w,\alpha}}$, possibly ramified over z_α.

This tells us very directly how we should picture BCMs: locally they are many disk coverings, of possibly different degrees, and in general being possibly infinite-to-one, in contrast to proper non-constant holomorphic maps. Moreover, branched covering maps $p : S \to T$ are quotient maps, being open, continuous and surjective.

Proposition 19.3. *If $p : S \to T$ is a branched covering map, the set of critical values is closed and discrete.*

Proof. Let $\hat{w}$ be a critical value of the regular branched covering $p : S \to T$. Let W be a BCM-neighborhood of w, such that $p^{-1}(W) = \bigsqcup_\alpha U_\alpha$. As stated previously, for each α, $p^{-1}(\hat{w}) \cap U_\alpha = \{\hat{z}_\alpha\}$, so that for $w \neq \hat{w}$ in W, $p^{-1}(w)$ is contained in $\bigsqcup_\alpha U_\alpha$ and has $d_{\alpha,\hat{w}}$ distinct element in each U_α, none of which is a critical point given that the branching is only over $\hat{z}_\alpha$. Hence $W \setminus \{w\}$ has no critical values.

Moreover, of $\hat{w}$ is not a critical value, the same reasoning as above shows that W can contian no critical values, so that the set of non-critical values is open. $\square$

A *regular branched covering* is one for which there exists a group $\Gamma \subseteq \mathrm{Aut}(S)$ of conformal automorphisms of S for which the orbits of Γ are exactly the fibers of $p : S \to T$. More explicitly, for $z \in S$, $\Gamma z = p^{-1}(p(z))$. The fibers, hence the orbits, are discrete. Since the action of Γ is transitive on the fibers, the numbers $d_{w,\alpha}$ depend only on w.

Proposition 19.4. *Γ acts properly discontinuously on S.*

Proof. If $S = \hat{\mathbb{C}}$, for any point $z \in \hat{\mathbb{C}}$, $\mathrm{Stab}(z)$ is finite. Moreover, we have a bijection between the orbit Γz and $\Gamma / \mathrm{Stab}(z)$. Since the orbit is finite, Γ must be a finite group of conformal automorphisms, and hence consist of only elliptic Möbius transformations. Evidently Γ acts properly discontinuously on S.

Assume S is hyperbolic. Suppose $K \subseteq S$ is compact such that for infinitely many distinct $\gamma \in \Gamma$, we have $\gamma K \cap K \neq \emptyset$. Then we get a sequence of group elements γ_n and points $z_n \in K$ such that $\gamma_n z_n \in K$. By taking subsequences, we may assume that $z_n \to z_\infty \in K$ and $\gamma_n z_n \to w_\infty \in K$.

As S is hyperbolic, Γ acts by isometries with respect to the Poincaré metric, so that Γ, viewed as a family of continuous maps on S, is uniformly equicontinuous. In fact,

$$d(w_\infty, \gamma_n z_\infty) \leq d(w_\infty, \gamma_n z_n) + d(\gamma_n z_n, \gamma_n z_\infty) = d(w_\infty, \gamma_n z_n) + d(z_n, z_\infty) \to 0,$$

so that $\gamma_n z_\infty \to w_\infty$. Since the orbits of Γ must be discrete, by taking another subsequence, we may assume that $\gamma_n z_\infty = w_\infty$ for all n. In particular, for $\sigma_n = \gamma_1^{-1}\gamma_n \in \Gamma$, we have $\sigma_n z_\infty = z_\infty$ for all n.

By Montel's theorem, some subsequence $\sigma_n \to \sigma_\infty$ converges locally uniformly, and in particular on a neighborhood around z_∞. If z is another point in this neighborhood, where $\sigma_n z \to \sigma_\infty z$, as the orbits have to be discrete, we see that for some subsequence $\sigma_n z = \sigma_\infty z$ for all n. By taking countably many points in this neighborhood of z_∞ and a diagonal subsequence, we may assume that for all n, σ_n is the identity on a countable set of points accumulating on z_∞. But this implies that the σ_n must be the identity, a contradiction.

If $S = \mathbb{C}$, all conformal automorphisms of $\mathbb{C}$ are of the form $az + b$. If $\gamma \in \Gamma$ is such that $\gamma z = az + b$ and $|a| \neq 1$, by possibly taking γ^{-1} we may assume $|a| < 1$. Then

$$\gamma^n z = a^n z + b(1 + a + a^2 + \ldots + a^{n-1}) = a^n z + b\frac{1 - a^n}{1 - a}$$

which converges to $b/(1-a)$ as $n \to \infty$. But since the orbits must be discrete, this cannot happen; so for all $\gamma \in \Gamma$, we must have $\gamma z = az + b$ with $|a| = 1$. (Rest of proof should follow for all euclidean cases...) □

If $p : S \to T$ is a regular branched covering, at least as sets, we may identify T and S/Γ. In fact, as $p : S \to T$ is a quotient map, we have a homeomorphism $T \cong S/\Gamma$, through which we can pullback the conformal structure of T to S/Γ.

Conversely, if $\Gamma \subseteq \mathrm{Aut}(S)$, what are the conditions on Γ for $p : S \to S/\Gamma$ to be a regular branched covering, and in particular, for S/Γ to be a Riemann surface in a canonical way? From the above, it is necessary that Γ acts properly discontinuously on S.

Proposition 19.5. *If Γ is a group of conformal automorphisms of S acting properly discontinuously, then S/Γ has a unique conformal structure such that the projection $p : S \to S/\Gamma$ is holomorphic. Moreover, this projection will be a branched covering map.*

Remarks from McMullen: A general (smooth n-dimensional) orbifold $\mathcal{O}$ is defined in the following way. We are given an underlying Hausdorff topological space X with the following data $(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)$, where

- $(U_\alpha)_\alpha$ is a base of open sets for the topology on X;
- V_α are open sets in $\mathbb{R}^n$;
- Γ_α is a finite group of diffeomorphisms of V_α;
- $\phi_\alpha : V_\alpha \to U_\alpha$ is a continuous maps whose fibers are the orbits of Γ_α, therefore inducing a homeomorphism $U_\alpha \cong V_\alpha/\Gamma_\alpha$.

Moreover, the following compatibility condition must be satisfied; when $U_\alpha \subset U_\beta$, we have an injective homomorphism $H_{\beta\alpha} : \Gamma_\alpha \to \Gamma_\beta$ and a smooth embedding $\phi_{\beta\alpha} : V_\alpha \to V_\beta$ such that:

- $\phi_{\beta\alpha}(\gamma z) = H_{\beta\alpha}(\gamma)\phi_\alpha(z)$;

- $\phi_\beta = \phi_{\beta\alpha} \circ \phi_\alpha$.

For a complex n-dimensional orbifold, smoothness assumptions are replaced by holomorphicity.

In the case of complex 1-manifolds, this notion in fact coincides with the prescription of a ramification index $N : X \to \mathbb{N}$ on a Riemann surface, specifying disk coverings around the singular points. In fact, quotients of the disk in this form have a canonical conformal structure, so that the orbifold inherits a conformal structure naturally.

If (S, ν) and (S', ν') are complex 1-orbifolds, a *holomorphic map* from (S, ν) to (S', ν') is a holomorphic map $f : S \to S'$ on the underlying Riemann surfaces such that, for all $z \in S$, we have the condition:

$$\nu'(f(z)) \text{ divides } n(f, z)\nu(z),$$

where $n(f, z)$ is the local degree of f at z.

This is equivalent to the following lifting property. There exists U_α and U'_α neighborhoods of z and $f(z)$, with charts $\phi_\alpha : V_\alpha \to U_\alpha$, $\phi'_\alpha : V'_\alpha \to U'_\alpha$, and a holomorphic map $g_\alpha : V_\alpha \to V'_\alpha$ such that

$$f \circ \phi_\alpha = \phi'_\alpha \circ g_\alpha.$$

In fact, g_α will be (conformally equivalent to) a branched covering of the disk $z \mapsto z^m$, where $m = n(f, z)\nu(z)/\nu'(f(z))$.

We say that $f : (S, \nu) \to (S', \nu')$ is an *orbifold covering map* if it is a branched covering map from S to S' satisfying $\nu'(f(z)) = n(f, z)\nu(z)$.

With these notions, we can actually define the orbifold associated to a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. See Chapter 19.

Problem (E-1. The complex plane with 2 ramified points)**.** (1) If $S = \mathbb{C}$ with ramification function satisfying $\nu(1) = \nu(-1) = 2$ and with no other ramified points, show that the map $z \mapsto \cos(2\pi z)$ provides a universal covering $\mathbb{C} \to (\mathbb{C}, \nu)$.

(2) Show that the Euler characteristic $\chi(\mathbb{C}, \nu)$ is zero, and the fundamental group $\pi_1(\mathbb{C}, \nu)$ consists of all transformations of the form $z \mapsto \pm z + n$ with $n \in \mathbb{Z}$.

Proof. The map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \cos(2\pi z)$ is holomorphic and surjective, and the critical points are of the form $\frac{1}{2}n$, for $n \in \mathbb{Z}$. The critical values are 1 and -1, corresponding respectively to the points $n \in \mathbb{Z}$ and $n + \frac{1}{2}$, for $n \in \mathbb{Z}$. It is easy (?) to check that f is indeed a branched covering map over ± 1, with local degree 2 at the critical points. In fact, f will be a regular branched covering, since the group of conformal automorphisms $z \mapsto \pm z + n$ acts transitively on all the critical points in the preimage of 1 and of -1, properly discontinuously. As $\mathbb{C}$ is simply connected, it is a universal (orbifold) covering.

The Euler characteristic of $(\mathbb{C}, \nu)$ is given by

$$\chi(\mathbb{C}, \nu) = \chi(\mathbb{C}) + \sum \left(\frac{1}{\nu(w)} - 1 \right) = 1 + \left(\frac{1}{2} - 1 \right) + \left(\frac{1}{2} - 1 \right) = 0.$$

The group of deck transformations of $\mathbb{C} \to (\mathbb{C}, \nu)$ contains $z \mapsto \pm z + n$, and must be equal to it, given that any conformal automorphism is of the form $az + b$, and such transformation must preserve both the integers and the half-integers. □

Problem (E-2. $\hat{\mathbb{C}}$ with 3 ramified points)**.** For $S = \hat{\mathbb{C}}$ with three ramified points $\nu(0) = \nu(1) = \nu(\infty) = 2$, show that the rational map

$$\pi(z) = \frac{-4z^2}{(z^2 - 1)^2}$$

provides a universal covering $\hat{\mathbb{C}} \to (\hat{\mathbb{C}}, \nu)$. Show that $\chi(\hat{\mathbb{C}}, \nu) = 1/2$, that the degree $\chi(\hat{\mathbb{C}})/\chi(\hat{\mathbb{C}}, \nu) = 4$, and that the fundamental group consists of all transformations $\gamma : z \mapsto \pm z^{\pm 1}$.

Proof. Note that π is a proper 4-sheeted branched covering of $\hat{\mathbb{C}}$, where $\pi(\infty) = 0$, $\pi(\pm 1) = \infty$ and

$$\begin{aligned} \pi'(z) = \frac{-8z(z^2-1)^2 + 4z^2 2(z^2-1)2z}{(z^2-1)^4} &= -8z(z^2-1)\frac{(z^2-1) - 2z^2}{(z^2-1)^4} \\ &= 8z\frac{z^2+1}{(z^2-1)^3}. \end{aligned}$$

Then, in $\mathbb{C}$, we have the finite critical points 0 and $\pm i$. Moreover,

$$\frac{1}{\pi(z)} = \frac{(z^2-1)^2}{-4z^2} = -\frac{(z^2-1)^2}{4z^2},$$

whose derivative is

$$\begin{aligned} -\frac{2(z^2-1)2z(4z^2) - 8z(z^2-1)^2}{16z^4} &= -8z(z^2-1)\frac{2z^2 - (z^2-1)}{16z^4} \\ &= -(z^2-1)\frac{z^2+1}{2z^3} = -\frac{z^4-1}{2z^3}. \end{aligned}$$

It is 0 at ± 1, so that these points, mapping to ∞, are also critical points. Since π has degree 4, the local degree at ± 1 is 2, mapping to ∞.

We also have

$$\pi\left(\frac{1}{z}\right) = \frac{-4(1/z)^2}{((1/z)^2-1)^2} = \frac{-4z^4/z^2}{z^4(1/z^2-1)^2} = \frac{-4z^2}{(1-z^2)^2}$$

whose derivative is

$$-\frac{8z(1-z^2)^2 - 4z^2 2(1-z^2)(-2z)}{(1-z^2)^4} = -z(1-z^2)\frac{8(1-z^2)+16z^2}{(1-z^2)^4}$$
$$= -8z\frac{1-z^2+2z^2}{(1-z^2)^3} = -8z\frac{z^2+1}{(1-z^2)^3}$$

so that ∞ is a critical point mapping to 0. This implies that the local degree of π at 0 and ∞ are both 2, mapping to 0.

Finally, since $\pi(\pm i) = 1$, and these are both critical points, the local degrees are also 2, mapping to 1.

To conclude that π is also a regular branched covering, we consider conformal automorphisms γ of $\hat{\mathbb{C}}$ that act transitively on the fibers of π. In particular, they must be deck transformations, satisfying $\pi \circ \gamma = \pi$. Hence γ maps $\{0,\infty\} \mapsto \{0,\infty\}$, $\{\pm 1\} \mapsto \{\pm 1\}$, and $\{\pm i\} \mapsto \{\pm i\}$. Since a Möbius transformation is completely determined by the action on 3 points, the entire group of deck transformations must be of the form $\gamma : z \mapsto \pm z^{\pm 1}$. It is easy to compute that $\chi(\hat{\mathbb{C}}, \nu) = 1/2$ from the usual formula, and the deck transformations correspond to the fundamental group of the orbifold. □

Problem (E-3. Bad orbifolds)**.** For $\hat{\mathbb{C}}$ with one ramified point, or with two ramified points with different ramification indices, show that there can be no universal covering surface.

Proof. Lemma E.2 affirms, among other things, that if $\widetilde{S_\nu}$ is the universal orbifold cover of (S,ν) with finite degree d, then $\chi(\widetilde{S_\nu}) = \chi(S,\nu)d$.

In the case of $\hat{\mathbb{C}}$ with only one ramified point, we have $\chi(\hat{\mathbb{C}},\nu) = 1 + \frac{1}{\nu(\hat{z})}$, if $\hat{z}$ is the ramified point. If the universalc covering has finite degree d, then

$$\chi(\widetilde{S_\nu}) = \left(1 + \frac{1}{n}\right) d = \frac{d(n+1)}{n},$$

which is an integer if and only if n divides d. In this case, the Euler characteristic is $\geq n+1 \geq 3$, which is impossible. Hence the covering must be of infinite degree, being either $\mathbb{C}$ or $\mathbb{D}$. But this is also a contradiction with Lemma E.3. A similar analysis is possible with the other case of a bad orbifold. (One may check that the fundamental group will need to have a non-integer amount of elements.) □

References

[1] C. McMullen, *Complex Dynamics and Renormalization*, Princeton University Press, 1995.

[2] J. Milnor, *Dynamics in One Complex Variable*, Vieweg, 2000.