Milnor on The Yoccoz Puzzle

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Comments for self on Milnor's exposition of the Yoccoz puzzle for local connectivity of polynomial Julia sets.

1 Local connectivity of quadratic Julia sets

1.1 Preliminaries

Let f be a polynomial. Recall the following theorems from [1]:

Theorem 1.1 (18.3. Landing Criterion). *For a polynomial connected Julia set J (that is, such that all of its critical orbits are bounded), J is locally connected if and only if the Böttcher conjugacy map near ∞ extends to a continuous surjective map $\varphi : \mathbb{R}/\mathbb{Z} \to J$. The values in J correspond to the landing points of external rays.*

The proof follows from the theory of Carathéodory ends.

Theorem 1.2 (18.5. Locally connected Julia sets). *If the polynomial Julia set J is connected and locally connected, then every periodic point in J is either repelling or parabolic. Moreover, every cycle of Siegel disks will contain a critical point on its boundary.*

In other words, if f is a polynomial, J is connected, and has either a Cremer periodic point or a cycle of Siegel disks without boundary critical point, then J is not locally connected.

The proof follows from the Snail lemma and compactness arguments on the set of external rays landing at the periodic (which can be assumed fixed) point. As for the Siegel disks, assuming local connectedness one may extend the linearization to the boundary homeomorphically, and repeat the compactness arguments for the rays landing on this boundary.

Theorem 1.3 (**18.10. Rational Rays Land**). *Every periodic external ray lands at a repelling or parabolic periodic point. If the angle is preperiodic, then it lands at a preperiodic point.*

The proof follows from the fact that rays landing at points must behave well with respect to images and preimages, and the set of rays landing at a specific point must preserve the cyclic ordering.

Theorem 1.4 (**18.11-13. Repelling and Parabolic points are landing points**). *If $z_0 \in J$ is a repelling periodic point, then at least one rational ray lands on z_0. Moreover, finitely may rays land on z_0, all with the same period.*

Suppose z_0 is a parabolic fixed point whose multiplier is a primite q-th root of unity, then every repelling petal of z_0 has at least one ray landing through it. All rays landing at z_0 are periodic with period q. We may extend the result to periodic parabolic points.

The proofs are considerably lengthier: the main idea is to consider the "linearized Julia set" at the point, as a subset of a suitable shift space with respect to the repelling behavior near it. The "linearized Fatou components" will be coverings of the basin at infinity and must eventually me mapped onto themselves; this is seen through detailed arguments from hyperbolic geometry.

1.2 First Part

With these preliminaries, let $f(z) = z^2 + c$ have a single critical point $c_0 = 0$ and two repelling fixed points α and β, so that the ray R_0 lands at β. There will be q rays landing at α, all having the same period, and being permuted among themselves by the doubling map. Later we will see that there is a single cycle of rays, hence of period q. Note that $q > 1$, since the only fixed ray is R_0. These rays landing at α will disconnect J into q pieces. As the critical point $c = 0$ belongs to one of the depth 0 puzzle pieces, it will follow the dynamics of theses rays around α, justifying the labeling according to it (we see later why). We will assume that the critical point is not preperiodic. All the depth 0 puzzle pieces are seen to be simply connected, as will be all other puzzle pieces (see also problem 1).

If P_0 is a depth 0 puzzle piece, and it does not contain the critical value $f(c_0)$, then $f^{-1}(P_0)$ consists of exactly two simply connected puzzle pieces mapping to it by a conformal isomorphism. Otherwise, $f^{-1}(P_0)$ is a single critical piece, simply connected and mapping to it by a 2-fold ramified covering. This inductively shows that all puzzle pieces are connected and simply connected.

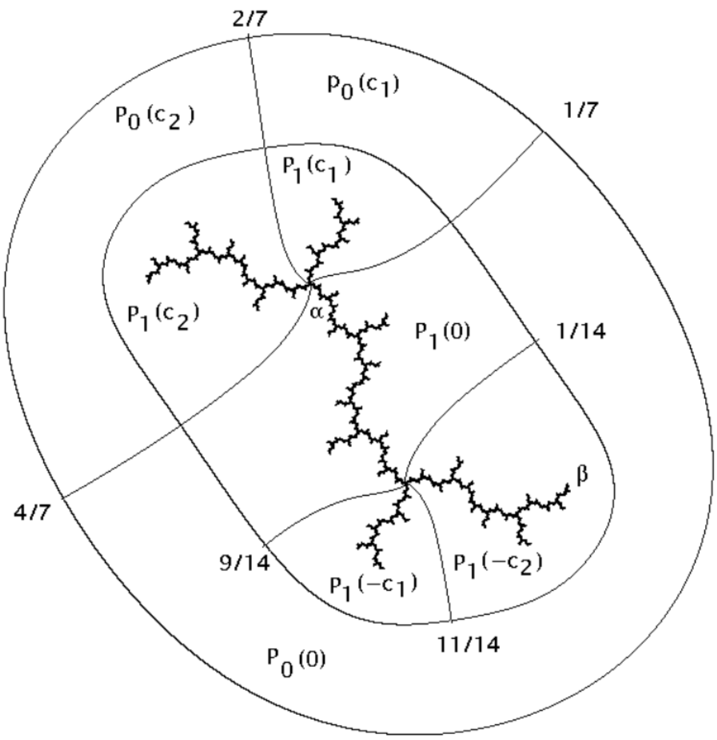


Figure 1: depth 1 Yoccoz puzzle for $z^2 + i$ taken from [1].

Naturally, the puzzle pieces of the same depth partition the neighborhood $\{G(z) < 2^{-d}\}$ of the Julia set J and have disjoint interiors. If q_d is the number of puzzle pieces of depth d, then $q_{d+1} = 2q_d + 1$, and

$$q_d = 2^d(q - 1) + 1.$$

We also see that if P_{d+1} is a puzzle piece which intersects P_d, then $P_{d+1} \subset P_d$. This is true for $d = 0$, and since $P_d = f^{-1}(Q_{d-1})$ and $P_{d+1} = f^{-1}(Q_d)$ for Q_{d-1} and Q_d other puzzle pieces, we have that

$$P_d \cap P_{d+1} = f^{-1}(Q_{d-1} \cap Q_d) = f^{-1}(Q_d) = P_{d+1},$$

since $f(z) \in Q_{d-1} \cap Q_d$ and, by induction, $Q_d \subset Q_{d-1}$.

On the proof of Lemma 1, part b): the hypothesis that the critical annulus $A_d(0)$ of depth d is excellent implies that we cannot have $d = d' - lk$ for some l, otherwise the north-east diagonal would "match up" at depth d and find a semicritical annulus at depth d and column $(l + 1)k$. This would contradict that the annulus is excellent.

We also cannot find a critical annulus at depth d through this north-east diagonal. Otherwise we would find a critical annulus at depth $d-1$ for some column of the interval $lk < i < (l+1)k$, which, from the tableau rules, had to have been copied over from the interval $0 < i < k$ at depth $d-1$. But this implies that in this interval of columns there is either a critical or sem-icritical annulus at depth d, contradicting that either $A_d(0)$ is excellent or that the column k is the first one at depth d we go right to find a critical annulus.

If we forego the hypothesis that $A_d(0)$ is excellent, we could possibly find a critical or semi-critical annulus at depth d in this diagonal. If it is a semi-critical annulus, we are still able to find the second child after going right towards the column m; but not in the other case.

Question: How do we know that the labeling of the depth 0 pieces behaves as predicted? That is, how do we know that each depth 0 contains is uniquely determined by which iterate c_i of the critical point it contains, for $0 \leq i \leq q-1$, assuming f is not postcritically finite? This depends on understanding the combinatorics of how the depth 0 pieces are permuted among themselves, which in turn depends on how the intervals between the angles behave under the doubling map.

First we recall that indeed we have $q > 1$ rays landing at α, and that they are all periodic with the same period. But do we know a priori that they are all in the same cycle of period q? This is not immediate from theorem 1.4; we shall soon prove this in the case of a repelling fixed point, instead of the more general situation of a periodic point. But for now, we assume this fact.

We know that locally at α, f is a local homeomorphism that preserves orientation, so not only are the angles $0 < \theta_0 < \ldots < \theta_{q-1}$ stable under doubling, they must rotate around the fixed point according to some rotation number p/q. More precisely, there must exist some $1 < p < q-1$ such that $2\theta_i \equiv \theta_{i+p}$, for all i. This also shows that certain combinatorial schemes of angles are not realizable by a ray portrait around a repelling fixed point, for example

$$\frac{1}{5} \mapsto \frac{2}{5} \mapsto \frac{4}{5} \mapsto \frac{3}{5} \mapsto \frac{1}{5}.$$

Let $P_0(c_0)$ be the depth 0 piece which contains the critical point. Any other depth 0 piece must map conformally onto its image, and given the rotating pattern of the rays, must cover a single other depth 0 piece. More precisely, if P is a piece corresponding to the interval $[\theta_i, \theta_{i+1}]$ of consecutive rays, the image must correspond to the interval of consecutive rays $[2\theta_i, 2\theta_{i+1}]$. This also shows that $\beta \in P_0(c_0)$, as it is the only piece that can map to itself.

Moreover, $-\alpha \in P_0(c_0)$, as near $-\alpha$ there are points that are mapped into all other pieces.

As the circle must cover itself twice under the angle doubling on the equipotentials, the critical depth 0 piece must cover the entire circle at least once, and if its interval is $[\theta_j, \theta_{j+1}]$, it must cover the piece corresponding to $[2\theta_j, 2\theta_{j+1}]$ twice. In particular, the interval $[\theta_j, \theta_{j+1}]$ must be the unique interval of length greater than $1/2$ and consequently the biggest interval.

If l is the length in radians of the smallest interval between consecutive angles, the lengths of all intervals are

$$l,\ 2l, 4l,\ \ldots,\ 2^{q-1}l,$$

so that $l = 1/(2^q - 1)$ and the interval of length $2^{q-1}/(2^q - 1)$ must correspond to $P_0(c_0)$. As the angles themselves must be periodic of period q under doubling, and therefore of the form $\theta_i = r_i/(2^q - 1)$, the smallest interval is of the form $[r/(2^q - 1), (r+1)/(2^q - 1)]$. Moreover, as $\beta \in P_0(c_0)$, being the landing point of the ray of angle 0, necessarily $P_0(c_0)$ is the piece corresponding to the interval $[\theta_{q-1}, \theta_0]$.

Given the rotation number p/q on the angles under doubing, we also see that we can label the depth 0 pieces according to the critical orbit c_i as long as $c_1 \notin P_0(c_0)$. Suppose by contradiction that were the case, and let Q_0 be the depth 0 piece that maps conformally onto $P_0(c_0)$ covering it. As c_1 has only the point c_0 in its preimage, we would need to have $c_0 \in Q_0$ and therefore $Q_0 = P_0(c_0)$, a contradiction with the conformal mapping. Hence $c_1 \notin P_0(c_0)$, and we can label the depth 0 according to the critical orbits. This also justifies the labeling of the $2q - 1$ depth 1 pieces as $P_1(c_i)$ and $P_1(-c_i)$.

We return to the question of there possibly existing m cycles of rays landing at α, each of period q, so that there are $q' = mq$ total rays. Given the local homeomorphism at α, we still have a rotation angle of p'/q', where given the labeling $0 < \theta_0 < \ldots < \theta_{q'-1} < 1$, $2\theta_i \equiv \theta_{i+p'}$.

If $P_0(c_0)$ is the critical piece, we still conclude that any other depth 0 maps conformally onto its image, covering a single other depth 0 piece given the rotation number, and $\beta \in P_0(c_0)$. In particular, the interval corresponding to $P_0(c_0)$ is $[\theta_{q'-1}, \theta_0]$.

By focusing on just one cycle of rays and forgeting the others, we are still led to the conclusion that the lengths of angle intervals for rays *in this cycle* are distributed as

$$l,\ 2l,\ 4l,\ \ldots,\ 2^{q-1}l,$$

where $P_0(c_0)$ is the piece corresponding to the biggest length. Therefore c_0 must be in this intersection of intervals of consecutive angles of the same cycle, all of length $2^{q-1}/(2^q - 1)$; in particular, the length of $[\theta_{q'-1}, \theta_0]$ must be less than $1/2$. But this is not possible if the piece $P_0(c_0)$ is to cover all other pieces, so that there must be a single cycle and $m = 1$.

Here's a shorter proof of the fact that there must be only one cycle of rays, due to Jeremy Kahn: if $[\theta_i, \theta_{i+1}]$ is an interval of consecutive angles, then the forward images of the corresponding piece must eventually contain the critical point, otherwise the length of the interval would be multiplied by a power of 2 before coming back to itself, which is not possible. As the critical piece must be contained in just one cycle of these angle intervals, there must be only one.

A further remark: most of these initial ideas generalize to $z^d + c$, where we have $d - 1$ fixed rays for the angles $j/(d-1)$, $0 \leq j \leq d-2$, and they all must correspond to distinct fixed points in J. Otherwise they would cut up J in halves which would map not univalently onto their images, hence containing a critical point. But since there is only one critical point not counting multiplicity, this is not possible. The fixed point α which is left must have a single cycle of rays landing on it, allowing for the definition of the Yoccoz puzzle from it.

1.3 Problems

Problem (1-1. Local Connectivity). Prove that the intersection of $J(f)$ with each puzzle piece is connected. Conclude that $J(f)$ is locally connected at z whenever $\bigcap P_d^*(z) = \{z\}$.

Proof. Let $d > 0$ be the smallest depth for which $P_d(z) \cap J$ is disconnected, where $P_d(z)$ is a puzzle piece. If $P_d(z)$ is not a critical piece, then $f : P_d(z) \to P_{d-1}(f(z))$ is a conformal isomorphism and $f : P_d(z) \cap J \to P_{d-1}(f(z)) \cap J$ a homeomorphism, so that $P_{d-1}(f(z)) \cap J$ would be disconnected and contradict our assumption on smallest depth. If $P_d(z)$ is a critical piece, then $f : P_d(z) \to P_{d-1}(f(z))$ is a ramified 2-covering, which implies that $P_d(z)$ is connected, but that $P_{d-1}(f(z)) \cap J$ is disconnected. More explicitly, If U and V are two components of $P_d(z)$, such that $0 \in U$, as each point has exactly two preimages counted with multiplicity, it cannot be the case that they both map to the same connected set. Similar arguments can be used to show that each puzzle piece is simply connected.

As $\bigcap P_d^*(z) = \{z\}$ and each puzzle piece is a neighborhood of z in J, since they are arbitrarily small, J would be locally connected at z. $\square$

Problem (1-2. Semicritical annuli). If $A_d(z)$ is a non-degenerate semi-critical annulus of depth $d > 0$, show that $A_d(z)$ is the union of

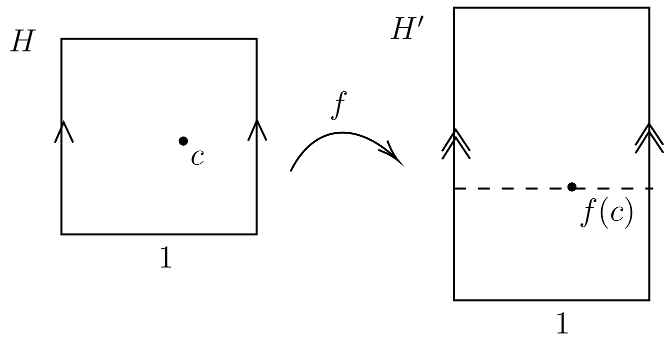
(1) a ramified two-fold covering of $A_{d-1}(f(z))$, and

(2) a conformal copy of $P_d(f(z))$.

Using Grötzsch inequality, prove that $\operatorname{mod} A_d(z) > \frac{1}{2} \operatorname{mod} A_{d-1}(f(z))$.

Proof. We have $A_d(z) = P_d(z) \setminus P_{d+1}(z)$, $P_d(z)$ is a ramified 2-covering of $P_{d-1}(f(z))$, and $P_{d+1}(z)$ is one of the two conformal copies of $P_d(f(z))$ which form $f^{-1}(P_d(f(z))) \subset P_d(z)$, since $P_d(f(z))$ does not contain the critical value. Hence $f^{-1}(A_{d-1}(f(z)))$ corresponds to $P_d(z)$ with two holes cut out, corresponding to the components of $f^{-1}(P_d(f(z)))$, one of which is $P_{d+1}(z)$. Gluing back in the other component, we obtain the semi-critical annulus $A_d(z)$. Note that the two-fold ramified covering of $A_{d-1}(f(z))$ is not an annulus, but as mentioned before, a disk with two holes.

Consider the picture below, where we have a 2-fold ramified covering $f : \mathcal{C} \to \mathcal{C}'$ of a cylinder of modulus H onto a cylinder of modulus H'. We cut up $\mathcal{C}'$ at the branch value into two straight subcylinders $\mathcal{C}_1'$ and $\mathcal{C}_2'$:



By considering the double unbranched covering $f|_{f^{-1}(\mathcal{C}_1')} : f^{-1}(\mathcal{C}_i') \to \mathcal{C}_i'$, we have two possibilities. Either $f^{-1}(\mathcal{C}_i')$ consists of two conformal copies of $\mathcal{C}_i'$, where one is essentially embedded in $\mathcal{C}$, or $\mathcal{C}_i'$ is doubly covered by an essentially embedded annulus in $\mathcal{C}$ of half the modulus. Whichever case happens for $\mathcal{C}_1'$, the other happens for $\mathcal{C}_2'$, so we may assume that the first case happens for $\mathcal{C}_1'$, and it has modulus h'.

By the inclusion of these annulli as preimages in $\mathcal{C}$, we obtain by Grotzsch's inequality also that

$$H \geq h' + \frac{1}{2}(H' - h') = \frac{H' + h'}{2} > \frac{1}{2}H',$$

since $h' > 0$. This concludes the modulus estimate. □

In some sense, the disk with two holes should have a "modulus" which is $\frac{1}{2} \operatorname{mod} A_{d-1}(f(z))$, given by the ramified 2-covering of $A_{d-1}(f(z))$, and the holomorphic inclusion of $f^{-1}(A_{d-1}(f(z))) \hookrightarrow A_d(z)$ should give the inequality between the moduli. This can be formalized through the language of extremal length (or extremal width), and how it behaves with respect to ramified coverings.

Recall that an annulus A is always conformally isomorphic to a cylinder $\mathcal{C}$ of height Δy and circumference Δx by identifying the opposite vertical sides on a rectangle $[0, \Delta x] \times [0, \Delta y]$, where the modulus is

$$\operatorname{mod}(A) = \operatorname{mod}(\mathcal{C}) = \frac{\Delta y}{\Delta x}.$$

More generally, given a family of locally rectifiable paths Γ on some Riemann surface S, the *extremal length* $\mathcal{L}(\Gamma)$ of Γ is given by

$$\mathcal{L}(\Gamma) = \sup_{\rho} \frac{\left(\inf_{\gamma \in \Gamma} L_\rho(\gamma)\right)^2}{A_\rho(S)},$$

where the supremum ranges over all conformal metrics on S whose area is $\neq 0, \infty$. The *extremal width* $\mathcal{W}(\Gamma)$ is

$$\mathcal{W}(\Gamma) = \inf_{\rho} \{A_\rho(S) \colon \forall \gamma \in \Gamma,\ L_\rho(\gamma) \geq 1\},$$

where now we consider the conformal metrics on S satisfying the above normalization condition on the paths in Γ. They are inverses of one another. As an example, we may take Γ to be the family of paths in the annulus A that connect the boundary components, so that $\mathcal{L}(\Gamma) = \operatorname{mod}(A)$.

Suppose $f : S \to T$ is a d-sheeted unramified holomorphic covering, and Γ is a path family on T. Let $f^*\Gamma$ be the path family on Γ consisting of all paths in S whose image under f is a path in Γ:

$$\gamma \in f^*\Gamma \iff f_*\gamma = f \circ \gamma \in \Gamma.$$

We show that

$$\mathcal{W}(f^*\Gamma) = d\mathcal{W}(\Gamma).$$

For this, let $\rho = \rho(z)|dz|$ be a conformal metric on T such that $L_\rho(\gamma) \geq 1$ for all $\gamma \in \Gamma$, and $f^*\rho$ the correspoding pullback metric on S. For $\widetilde{\gamma} \in f^*\Gamma$ a lift of γ,

$$L_{f^*\rho}(\gamma) = \int_\gamma f^*\rho = \int_{f\circ\gamma} \rho = L_\rho(f_*\gamma) \geq 1,$$

as $f : (S, f^*\rho) \to (T, \rho)$ is a local isometry, preserving the length of curves. Moreover, by taking a covering of T by evenly covered neighborhoods and a partition of unity subordinate to it, and pulling back these neighborhoods to S, we find that $A_{f^*\rho}(S) = d \cdot A_\rho(T)$. Hence

$$\mathcal{W}(f^*\Gamma) \leq d\mathcal{W}(\Gamma).$$

Now let ν be a conformal metric on S such that $L_\nu(f^*\gamma) \geq 1$ for all $\gamma \in \Gamma$), and define the metric $\hat{\nu}$ on T as the pointwise sum on the fibers:

$$\hat{\nu}(w) := \sum_{z \in f^{-1}(w)} \nu(z).$$

We can see that $L_{\hat{\nu}}(\gamma) \geq d$ for $\gamma \in \Gamma$, being equal to the sum of the lengths of all lifts of γ. By taking the metric $d^{-1}\hat{\nu}$ on T, and since $A(\hat{\nu}) = A(\nu)$, we get

$$A(d^{-1}\hat{\nu}) = d^{-1}A(\nu).$$

As $L_{d^{-1}\hat{\nu}}(\gamma) \geq 1$, by taking the relevant infimums we obtain $\mathcal{W}(f^*\Gamma) \geq d\mathcal{W}(\Gamma)$, which concludes the desired equality.

Problem (1-3. Non-degenerate annuli). Show that an annulus $A_d(z_m)$ is non-degenerate if and only if the corresponding annulus $A_0(z_{d+m})$ of depth 0 is semi-critical.

Proof. The result is true for $d = 1$. The semi-critical annuli for $d = 0$ are those whose inner regions are $P_1(-c_i)$, for $c_i \neq 0$. (Does this depend on the fact that 0, $-\alpha$ and β all belong to the same depth 0 piece?) Being a non-degenerate annulus is a property preserved by the pre-images and images, so that the result will be true for any depth. □

Problem (1-4. Further Tableau Rules.). Let $q \geq 2$ be the number of external rays landing at the fixed point α. Show that the semi-critical depth of a tableau column can never take the values $1, \ldots, q-1$. Show that at most $q-1$ consecutive columns can be completely off-critical (semi-critical depth -1), and show that $\operatorname{scd}(z_i) = -1$ for $m < i < m+q$ if and only if the m-th column has semi-critical depth $\operatorname{scd}(z_m) \geq q$.

Proof. Recall that for depth 0, we have the q puzzle pieces $P_0(c_0), \ldots, P_0(c_{q-1})$, corresponding to the regions in between the external rays. For depth 1, we also have the pieces $P_1(-c_i) \subset P_0(0)$.

Suppose some column has semi-critical depth d for $1 \leq d \leq q-1$, so that for this point $\hat{z}$ we have $\hat{z} \in A_d(0) = P_d(0) \setminus P_{d+1}(0)$. Therefore

$$f^k(\hat{z}) \in P_{d-k}(c_k),$$

and in particular, $f^d(\hat{z}) \in P_0(c_d)$. Since for $1 \le d \le q-1$, we have that $J(f) \cap P_0(c_d) = P_1(c_d)$, we get $f^d(\hat{z}) \in P_1(c_d)$, and $f^{d+1}(\hat{z}) \in P_0(c_{d+1})$.

Letting $\hat{z}_k := f^k(\hat{z})$, we have $\hat{z}_d \in P_1(c_d)$ and $\hat{z}_{d-1} \in P_1(c_{d-1})$ from the above. For $d > 1$, we have two puzzle pieces as the preimage $f^{-1}(P_1(c_d))$, one of which is $P_2(c_{d-1})$. Each one is contained in one of two distinct puzzle pieces of $f^{-1}(P_0(c_d))$, one being $P_1(c_{d-1})$. And since $\hat{z}_{d-1} \in P_1(c_{d-1})$, we will also conclude that $\hat{z}_{d-1} \in P_2(c_{d-1})$.

We proceed inductively until we obtain that $\hat{z}_1 \in P_d(c_1)$, where c_1 is the critical value. Here, $P_d(c_1)$ actually has only one puzzle piece as its preimage, being $P_{d+1}(0)$, and $\hat{z}$ must belong to it. But this will give us a contradiction with $\hat{z} \in A_d(0)$.

If a column is completely off-critical, this implies that $\hat{z} \in P_0(c_i)$ for some $1 \le i \le q-1$. But in q iterations, at least one iterate must map into $P_0(0)$, giving a semi-critical depth of at least 0. Combining the two facts above, we must have that the semi-critical depth of z_m must be $\ge q$.

□

Problem (1-5. The critical orbit is generically dense). It is convenient to say that a property of certain points in a compact set is generically true if its true throughout a countable intersection of dense open subsets. For example, one can show that for generic c in the boundary of the Mandelbrot set, the map f_c is not renormalizable, with both fixed points repelling. Let $U_d \subset \partial M$ be the set of parameter values c in the boundary of the Mandelbrot set such that every puzzle piece of depth d for f_c contains a post-critical point $c_i = f_c^i(0)$. Show that U_d contains a dense relatively open subset of ∂M. (To prove density, use the fact that periodic points are dense in $J(f_c)$, and use Montel's theorem.) For a generic parameter value $c \in \partial M$, conclude that the closure of the critical orbit is the entire Julia set $J(f_c)$. Conclude also that no non-degenerate annulus can be excellent in the generic case.

Proof. Recall that α and β move smoothly with respect to c, along with any external ray, given that the Böttcher uniformization of $\mathbb{C} \setminus K(f_c)$ is holomorphic in c. In fact, more is true. Given a repelling periodic point z_0 for f_{c_0}, for all c in a neighborhood of c_0, z_0 can be continued analytically as a periodic point of the same period $z(c)$, and if $\{\theta_i\}$ is the set of angles of the rays landing at z_0, $\{\theta_i\}$ is also the set of angles of the rays landing at $z(c)$. An analogous result holds true if z_0 is preperiodic, provided it and no point in its forward orbit is c_0 ([2]).

With this, for $c_0 \in \partial M$ such that the critical orbit $f^i(c_0) = c_i$ does not hit the fixed point $\alpha(c_0)$, let V be some neighborhood of ∂M in $\mathbb{C}$. Since we only consider the Yoccoz puzzle up to depth d, we may shrink V around c_0

and assume that, for all $c \in V$, the Yoccoz puzzle for f_c is combinatorially the same, since the finitely many preimages of α will vary holomorphically with c, along with the rays landing on them and the equipotentials $G_c(z)$. Hence we can make sense of a Yoccoz puzzle of depth d even if $c \in V \setminus M$.

Suppose that c_0 is such that the critical orbit passes through the interior of all depth d puzzle pieces. As we only need to keep track of finitely many iterates, and they move holomorphically with respect to c, for a small neighborhood W of c_0 we guarantee that the critical orbit of $c \in W$ still passes through all depth d puzzle pieces (for f_c). This implies that c_0 is in the interior of U_d.

As a consequence of topological transitivity and density of repelling periodic points on J_c, we can show that there exists a repelling periodic point $p(c_0)$ whose orbit passes through the interior of all depth d pieces for c_0. There are in fact inifintely many such points, and we may choose one whose orbit does not intersect the grand orbit of c_0. Let $p_1(c_0)$ and $p_2(c_0)$ non-periodic points in the grand orbit of $p(c_0)$, so that they are preperiodic and mapped into the cycle of $p(c_0)$. By our assumptions, we may shrink V so as to assume that $p_i(c)$ moves holomorphically with respect to $c \in V$ for $i = 1, 2$. Given the family of holomorphic functions $\varphi_n : V \to \hat{\mathbb{C}}$ given by $\varphi_n(c) = f_c^n(c)$, we also construct

$$g_n(c) = \frac{(\varphi_n(c) - p_1(c))(c - p_2(c))}{(\varphi_n(c) - p_2(c))(c - p_1(c))}.$$

If we assume that, for all $c \in V$, the critical orbit does not pass through all the depth d pieces, then the functions g_n avoid $0, 1$ and ∞ in its image, so that it becomes a normal family. This implies that φ_n is also normal, and there exists a subsequence $\varphi_{n_k}(c) = f_c^{n_k}(c)$ and a holomorphic function $\varphi : V \to \hat{\mathbb{C}}$ such that $\varphi_{n_k} \to \varphi$ locally uniformly.

For $c \in V \setminus M$, we see that $f_c^n(c) \to \infty$, so that φ_{n_k} must convergence uniformly to ∞. But this cannot happen for $c \in \partial M$, since the critical orbit is contained in the filled Julia set, a contradiction. Hence, there must exist some $c' \in V$ such that the critical orbit of $f_{c'}$ passes through all the depth d pieces, or more precisely, that some iterate of c' maps to $p(c')$ or $f(p(c'))$. Moreover, since then the critical orbit for c' is preperiodic, we in fact have that $c' \in \partial M$. This concludes the density part of the argument.

We see that U_d contains a dense open set of ∂M, so that $\bigcap_d U_d$ is dense in ∂M, corresponding to a generic set of points. This implies that for generic c, any puzzle piece of any depth contains a critical point. As we are also in the generic case where J is locally connected, the nested intersection of puzzle pieces reduces to points. This proves that the post-critical set accumulates at every point of J.

$\square$

References

[1] J. Milnor, *Dynamics in One Complex Variable*, Vieweg, 2000.

[2] D. Schleicher, *Rational parameter rays of the Mandelbrot set* (1998).