Critically periodic polynomial-like Thurston maps

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1 Introduction

Let $f : S^2 \to S^2$ be an orientation preserving degree d branched covering map of a topological 2-sphere, and P_f be its postcritical set. We assume P_f is finite. An important question is whether such a map is *Thurston equivalent* to a postcritically finite rational map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, that is, there exists homeomorphisms $\theta, \theta' : S^2 \to \hat{\mathbb{C}}$ such that the diagram below commutes:

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\theta} & (\hat{\mathbb{C}}, P_g) \\ \downarrow f & & \downarrow g \\ (S^2, P_f) & \xrightarrow{\theta'} & (\hat{\mathbb{C}}, P_g) \end{array}$$

and θ is isotopic to θ' relative to P_f. Thurston gave a necessary and suficcient topological condition on whether such an equivalence exists. Let Γ be a set of disjoint, non-trivial (non-nullhomotopic and non-peripheral) simple closed curves on $S^2 \setminus P_f$, no two homotopic to each other. We also say that Γ is a multicurve. For each $\gamma \in \Gamma$, the pullback $f^{-1}(\gamma)$ is a set of disjoint simple closed curves, mapping to γ with some degree. We say that Γ is *f-stable* if for all $\gamma \in \Gamma$, every component of $f^{-1}(\gamma)$ is homotopic rel P_f to some curve in Γ. If $\Gamma = \{\gamma_j\}_j$, by considering the vector space $\mathbb{R}^\Gamma$, we have an induced linear pullback map $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ given on each basis element γ_j by

$$f_\Gamma(\gamma_j) = \sum_j \sum_\alpha \frac{1}{d_{\alpha,i,j}} \gamma_i,$$

where α ranges over all components γ_α of $f^{-1}(\gamma_j)$ which are homotopic to γ_i rel P_f, and $d_{\alpha,i,j}$ is the mapping degree of γ_α onto γ_j. Note that in this calculation we disconsider those components of the pullback which are peripheral. Moreover, being a matrix with all entries non-negative, there exists a real eigenvalue $\lambda(f,\Gamma)$ of f_Γ of largest absolute value. We also remark that this pullback map may also be defined for multicurves which are not necessarily f-stable, but as expected, information of non-trivial pullbacks outside of Γ is not recorded in f_Γ.

If f has hyperbolic orbifold (a condition visible from the portrait of f, that is, the information about $\Omega_f \cup P_f$ and the local degrees of the critical points), then f is Thurston-equivalent to a rational map if and only if $\lambda(f,\Gamma) < 1$. An f-stable multicurve with $\lambda(f,\Gamma) \geq 1$ is said to be an *obstruction.*

We restrict ourselves to polynomial-like mappings, that is, those for which there exists a fixed critical point of local degree d, so that f induces a proper degree d branched cover of the plane $\mathbb{R}^2$ onto itself. Moreover, we consider the additional hypothesis that all critical points of f are periodic, and we wish to prove the following:

Theorem 1.1. *Polynomial-like, critically periodic Thurston maps are unobstructed.*

2 Topology of branched covers and curves

Given two disjoint simple closed curves γ and η in $\mathbb{R}^2$, they are either separated or nested, as a consequence of the Jordan curve theorem. Equivalently, if D_γ denotes the open disk in $\mathbb{R}^2$ having γ as its boundary, we either have $D_\gamma \cap D_\eta = \emptyset$ or $D_\gamma \subset\subset D_\eta$ compactly contained (or the opposite inclusion). Given γ, we want to understand the compact set $f^{-1}(\overline{D_\gamma})$. It is the disjoint union

$$f^{-1}(\overline{D_\gamma}) = f^{-1}(\gamma) \sqcup f^{-1}(D_\gamma),$$

where $f^{-1}(D_\gamma)$ is open. Because f is open, we also have $f^{-1}(\overline{D_\gamma}) = \overline{f^{-1}(D_\gamma)}$, and as f is proper, both restricted maps

$$f|_{f^{-1}(\overline{D_\gamma})} : f^{-1}(\overline{D_\gamma}) \to \overline{D_\gamma}, \qquad f|_{f^{-1}(D_\gamma)} : f^{-1}(D_\gamma) \to D_\gamma$$

are also proper. Moreover, since f is a branched cover, $f^{-1}(D_\gamma)$ has finitely many open components. If U is one of these components, then $f|_U : U \to D_\gamma$ is also proper, which can be checked readily from the above facts. In particular, $\partial U \subseteq \partial f^{-1}(D_\gamma) = f^{-1}(\gamma)$, and from a general converse we obtain

$$f^{-1}(\gamma) = \bigcup_{i=1}^{m} \partial U_i.$$

Note that, in principle, ∂U_i could be disconnected. Suppose that for some $i \neq j$ we had $x \in \partial U_i \cap \partial U_j$. We may take small neighborhoods V of $f(x)$ and W of x such that W maps to V homeomorphically, since $f(x) \notin P_f$. If we assume the curve γ to be C^1 in $\mathbb{R}$ so that, by taking a small enough V, the intersection $D_\gamma \cap V$ is connected, the corresponding intersection in W is also connected, so that there is a unique component U_i of $f^{-1}(D_\gamma)$ intersecting W and for which $x \in \partial U_i$. This shows that boundaries of distinct components cannot intersect, and due to connectedness of each curve in $f^{-1}(\gamma)$, each one is contained in some ∂U_i.

Suppose U is some component of $f^{-1}(D_\gamma)$, whose boundary curves are organized by their *nesting depth*. Curves in a multicurve Γ which are not nested inside of another one have depth 0; we inductively define the depth of nesting for the other curves. Naturally curves having the same depth within Γ are separated. If there were more than one curve in $\partial U = \Gamma$ of depth 0, either U would be unbounded in $\mathbb{R}^2$ or disconnected, a contradiction. Hence ∂U consists of a γ' of depth 0 nesting some curves $\gamma'_1, \ldots, \gamma'_k$ of depth 1; topologically this corresponds to a disk with k holes. Since $f|_{f^{-1}(S^2 \setminus \overline{D_\gamma})} : f^{-1}(S^2 \setminus \overline{D_\gamma}) \to S^2 \setminus \overline{D_\gamma}$ is also proper, each disk $D_{\gamma'_j}$ would contain a component of $f^{-1}(S^2 \setminus \overline{D_\gamma})$, and hence contain a point in the preimage of $f^{-1}(\infty)$, a contradiction with the map being polynomial-like. This implies that $D_{\gamma'} = U_i$, and ∂U_i is a single component of $f^{-1}(\gamma)$. We conclude:

Lemma 2.1. *The components of $f^{-1}(\gamma)$ are all separated, and each component $\gamma' \subseteq f^{-1}(\gamma)$ bounds a disk $D_{\gamma'}$ such that $f|_{D_{\gamma'}} : D|_{\gamma'} \to D|_\gamma$ is a proper map of the same degree as $f : \gamma' \to \gamma$.*

We also obtain the following results:

Lemma 2.2. *If γ and η are two separated curves, and γ' and η' are components of $f^{-1}(\gamma)$ and $f^{-1}(\eta)$ respectively, then γ' and η' are separated.*

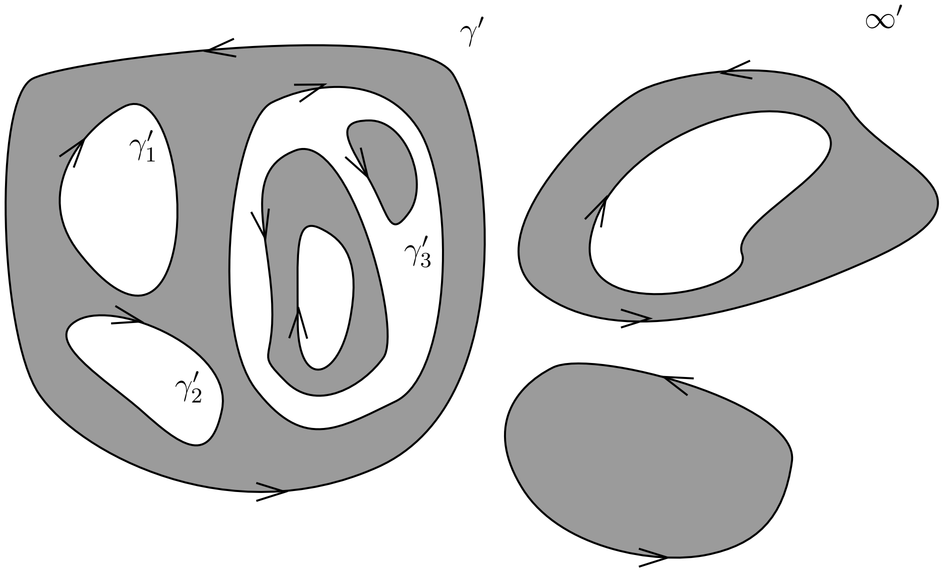
Proof. Suppose on the contrary that $D_{\gamma'} \subset D_{\eta'}$. Since f maps $D_{\gamma'}$ onto D_γ and $D_{\eta'}$ onto D_η, this would imply that there $D_\gamma \subset D_\eta$, a contradiction. □

Lemma 2.3. *Suppose that γ and η are two nested curves, with $D_\gamma \subset D_\eta$. Then for each component γ of $f^{-1}(\gamma)$, there is some component η' of $f^{-1}(\eta)$ such that γ' is nested inside of η'.*

Proof. Since $D_\gamma \subset D_\eta$, $f^{-1}(D_\eta)$ must intersect $D_{\gamma'}$, And by connectedness of $D_{\gamma'}$ it must be contained in some component of $f^{-1}(D_\eta)$. □

We may generalize the ideas above in the following form. Let $f : S^2 \to S^2$ be a topological branched cover, not necessarily post-critically finite, $\gamma \subset S^2 \setminus V_f$ a simple closed curve on the complement of the critical values of f, and D one of the two open disks that it bounds. We pick a point on $S^2 \setminus \overline{D}$ and label it ∞, and also pick a point $\infty' \in f^{-1}(\infty)$. Given any component U of $f^{-1}(D) \subset S^2 \setminus \infty' \cong \mathbb{R}^2$, we may again order ∂U by depth of nesting. We conclude that U will be the region bounded between one curve γ' of depth 0 (with respect to ∂U, **not** $f^{-1}(\gamma)$) and k separated curves $\gamma'_1, \ldots, \gamma'_k$ of depth 1.

We now consider the full multicurve $f^{-1}(\gamma) \subset \mathbb{R}^2 \setminus \infty' \cong \mathbb{R}^2$, but normalize the nesting depth to assume that γ' has depth 0, allowing for negative nesting depth. This just corresponds to translating it by some fixed value. For any $j = 1, \ldots, k$, we can similarly show that the region between γ'_j and the curves of depth 2 that it nests corresponds exactly to a component of $f^{-1}(S^2 \setminus \overline{D})$. Continuing for all depths, positive and negative, we can completely picture all components of $f^{-1}(D)$ and $f^{-1}(S^2 \setminus \overline{D})$. If we imbue γ with a given orientation, each component of $f^{-1}(\gamma)$ will have an associated orientation with which it maps to γ, and this orientation scheme is completely prescribed by the set of curves with lowest nesting depth. This is because each curve it nests will be transversed with opposite orientation to match.



Moreover, by Riemann-Hurwitz, if we know the degrees of the map f restricted to each region, we may also recover the number of critical points in each of them counting multiplicity. If U is a component of $f^{-1}(D)$, then $\chi(U) = 1 - k$, where k is again the number of "holes" in U. If b denotes the number of branch points in U counting multiplicity and d' the degree of $f|_U$, then

$$b = d'\chi(D) - \chi(U) = d' - k + 1.$$

(Here we remark that, for example, 0 is a critical point of multiplicity 1 for $z \mapsto z^2$, being a simple critical point, even though it's ramification index is 2.) To illustrate, if $d' = 1$, then $f|_U$ must be a homeomorphism and $k = 0$. If d_j is the degree of $f|_{\gamma'_j} : \gamma'_j \to \gamma$, where $\gamma'_0 = \gamma'$, then $d' = \sum_{j=0}^{k} d_j \geq k+1$, which implies that $b \geq 2k$. There are other inequalities possible, coming from the fact that there are exactly $2d - 2$ critical points counting multiplicity, and that $k + 1 \leq d' \leq d$, also because each of the k holes and the outside region of U must contain a preimage of ∞.

We go further and claim that any such picture, with associated colorings and mapping degrees, is realizable by a topological branched cover. More precisely:

Theorem 2.4. *Let $\Gamma \subseteq S^2$ be a finite collection of simple closed curves, $\deg : \Gamma \to \{1, 2, \ldots\}$ a given function, and $c : \pi_0(S^2 \setminus \Gamma) \to \{0, 1\}$ a coloring of the complentary regions of Γ such that no two adjacent regions share the same color. Then there exists a topological branched cover $f : S^2 \to S^2$ and a simple closed curve $\gamma \subset S^2 \setminus V_f$ such that $f^{-1}(\gamma) = \Gamma$, each $\gamma' \in \Gamma$ is mapped to γ with degree $\deg(\gamma')$, and if D_0, D_1 are the two disks bounded by γ, then $f^{-1}(D_i) = c^{-1}(i)$ for $i \in \{0, 1\}$.*

Proof. We first show that if $\overline{D}$ is a topological closed disk and $\overline{S}$ is a topological disk with k holes, where each of the $k + 1$ boundary components of $\overline{S}$ has an associated degree, then there exists a topological branched cover $g : \overline{S} \to \overline{D}$ with corresponding mapping degrees on the boundary components of $\overline{S}$. Let γ'_j be the boundary curves of $\overline{S}$, for $j = 0, \ldots, k$. For a topological model, it suffices to consider a rational function $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ whose only poles are one of order $\deg(\gamma_0)$ at ∞, and k finite distinct poles of order $\deg(\gamma'_j)$, for $j = 1, \ldots, k$. By taking a simple closed curve γ sufficiently close to ∞ in the spherical metric, we may identify the closure of the bounded component of γ with $\overline{D}$, so that $h^{-1}(\overline{D})$ will consist of a single component which is a closed disk with k holes. Moreover, the mapping degrees of the boundary components of $h^{-1}(\overline{D})$ will match the prescribed ones (also with an associated orientation).

To produce the full picture in the sphere, we only need to recreate the topological model for all regions in $S^2 \setminus \Gamma$, where those with different color must be created with opposite orientation, mapping to complementary disks on S^2. By gluing them together appropriately at the boundaries, and possibly post-composing the resulting map with an inversion to match the coloring, we obtain the result. ∎

It is a different matter to ask whether such pictures are always realizable for *post-critically finite* branched covers, and such that γ is non-trivial (or even belongs to an obstruction); we would need to keep track of P_f closely by deforming the glued maps in each region appropriately. We avoid answering this question here.

3 Dynamics

Let p_γ be the number of postcritical points inside of D_γ, and q_γ to be number of postcritical points outside of γ (Here $p_\gamma + q_\gamma = p - 1$, where we are not considering the point at infinity). Note that this number is a homotopy invariant in $\mathbb{R}^2 \setminus P_f$, and in order for γ to be non-peripheral, we must have $p_\gamma \geq 2$, $q_\gamma \geq 1$. Recall our assumption that f is critically periodic; this implies that $\Omega_f \subseteq P_f$ and that $f|_{P_f} : P_f \to P_f$ is injective, since P_f is distributed into the cyclic orbits of the critical points.

If γ' is a component of $f^{-1}(\gamma)$, then

$$p_{\gamma'} \leq p_\gamma,$$

since each point in $P_f \cap D_{\gamma'}$ has to map injectively to a point in $P_f \cap D_\gamma$. In other words, the number of postcritical points inside a curve is non-increasing under pullback. Furthermore, since the components of the pullback $f^{-1}(\gamma)$ must be separated, in particular they cannot be homotopic in $\mathbb{R}^2 \setminus P_f$ unless they are peripheral.

It's important to note, however, that homotopy classes in $\mathbb{R}^2 \setminus P_f$ may collapse together under pullback. More precisely, if γ is nested inside η in $\mathbb{R}^2 \setminus P_f$, and γ' is a component of $f^{-1}(\gamma)$ and η' is the unique component of $f^{-1}(\eta)$ which nests γ', it could be the case that γ' and η' are homotopic in $\mathbb{R}^2 \setminus P_f$ even if γ and η are not homotopic; This is because our actual mapping is $\mathbb{R}^2 \setminus f^{-1}(P_f) \to \mathbb{R}^2 \setminus P_f$, so that if x is a postcritical point which is inside η but outside γ, its corresponding preimage x' inside of η' and outside of γ' may not be a postcritical point. Another possible way to view this situation is through the composition of the two maps

$$\iota : \mathbb{R}^2 \setminus f^{-1}(P_f) \to \mathbb{R}^2 \setminus P_f, \qquad f : \mathbb{R}^2 \setminus f^{-1}(P_f) \to \mathbb{R}^2 \setminus P_f$$

which determine the dynamics.

Let Γ be an f-stable multicurve. For $\gamma_i, \gamma_j \in \Gamma$, there can be at most one component γ_α of $f^{-1}(\gamma_j)$ which is homotopic to γ_i in $\mathbb{R}^2 \setminus P_f$, given that the components of the pullback are separated; so the sum in $f_\Gamma(\gamma_j)$ has at most one term coming from the components α for each j. We define d_{ij} to be the degree of the mapping $f : \gamma_\alpha \to \gamma_j$, and $d_{ij} = 0$ if such γ_α homotopic to γ_i does not exist. Note that $\sum_i d_{ij} \leq d$, and that the non-zero entries of f_Γ are $1/d_{ij}$.

Fix some $\gamma_i \in \Gamma$, and suppose that γ_j and γ_k are separated curves in Γ such that there are components γ_α of $f^{-1}(\gamma_j)$ and γ_β of $f^{-1}(\gamma_k)$ where γ_α and γ_β are homotopic to γ_i (recall that this homotopy is in $\mathbb{R}^2 \setminus P_f$, and that these components are unique being homotopic to γ_i). As γ_i is non-trivial, this implies that γ_α and γ_β are nested, but this is only possible if γ_j and γ_k are nested or homotopic, which is a contradiction; hence either $d_{ij} = 0$ or $d_{ik} = 0$. In other words, among separated curves $\gamma_{j_1}, \gamma_{j_2}, \ldots, \gamma_{j_m} \in \Gamma$, at most one can have $d_{ij_l} \neq 0$.

This suggests that we should organize the multicurve Γ into a structure which guarantees that some curves are separated. One way to do so is by the values p_γ; naturally curves in Γ having the same number of postcritical points enclosed inside them must be separated. Moreover, with the hypothesis that the critical orbits are all periodic, the number p_γ is non-increasing under pullback, which further dictates how the pullback acts on this structure of Γ.

We consider a different structure of Γ, that of nesting. The depth n_γ is non-increasing under pullback, since separated curves stay separated and nested curves stay nested, but possibly collapsing to the same homotopy class in $\mathbb{R}^2 \setminus P_f$. Evidently curves having the same depth of nesting are separated.

If we order the curves in Γ by their depth of nesting, the matrix for f_Γ becomes block upper triangular, since a curve γ_i may be pulled back only to curves γ_j having depth of nesting $n_{\gamma_j} \leq n_{\gamma_i}$. Because the eigenvalues of a block upper triangular matrix are the collection of eigenvalues of each diagonal block, if Γ is an obstruction then for some nesting depth n the curves Γ_n having this depth can be extended to form an obstruction. By this reasoning, we may assume from the outset that our f-stable multicurve Γ which is an obstruction consists of only separated curves.

In this situation, by the previous reasonings, every row of f_Γ can have at most one non-zero entry. In other words, if γ_i is the pullback (or more precisely, homotopic to a pullback) of a curve in Γ, it is the pullback of a unique such curve γ_j; we may represent this as a formal mapping $\gamma_i \to \gamma_j$. If γ_i is not the pullback of any curve in Γ, so that the i-th row in f_Γ is identically zero, then any non-zero eigenvalue of f_Γ has to come from the minor matrix obtained from f_Γ by excluding the i-th row and the i-th column. Hence we may assume that no row is identically zero, as $\Gamma \setminus \{\gamma_i\}$ is still f-stable and constitutes an obstruction.

We therefore have a well-defined formal mapping $\hat{f} : \Gamma \to \Gamma$ of the curves as above. In principle, it need not be injective or surjective, as would be the case, for example, if a single curve $\gamma \in \Gamma$ has pullbacks homotopic to all curves in Γ and all other curves have trivial pullbacks, so that f_Γ has a single non-zero column. We want to extract from this dynamics of curves a *Levy cycle*. If f is a postcritically finite topological branched cover, a multicurve $\{\gamma_1, \ldots, \gamma_m\}$ is a *Levy cycle* if for all $j = 1, \ldots, m$, the curve γ_{j-1} is homotopic to a component γ'_j of $F^{-1}(\gamma_j)$ rel P_f, and $F|_{\gamma'_j} : \gamma'_j \to \gamma_j$ is a homeomorphism, that is, is of degree 1. It can be shown readily that any Levy cycle may be completed to a Thurston obstruction ([2]).

In our Thurston obstruction Γ, curves are either periodic under $\hat{f}$ or preperiodic. We may organize f_Γ as a block square matrix according to the periodic cycles, and the preperiodic curves. Moreover, note that $(f)^k_\Gamma = (f_\Gamma)^k$ and $\widehat{f^k} = \hat{f}^k$. By taking a high enough iterate of f, so that all periodic curves under $\hat{f}$ are fixed by f^k and all preperiodic curves are mapped into the set of periodic ones, we get that f^k_Γ has upper triangular block form

$$\begin{bmatrix} D & 0 \\ A & 0 \end{bmatrix}$$

where D is diagonal. It is sufficient then to consider the eigenvalues of D, that is, its diagonal entries; but each is a product of factors of the form $1/d_{ij}$ for curves in a cycle under $\hat{f}$. Then Γ is an obstruction only if at least one of these cycles under $\hat{f}$ consists entirely of degree 1 mappings, that is, a Levy cycle. Hence:

Theorem 3.1. *A polynomial-like postcritically finite topological branched cover $f : \mathbb{R}^2 \to \mathbb{R}^2$ has a Thurston obstruction if and only if it has a Levy cycle.*

Moreover, by the arguments preceding Lemma 2.1, this Levy cycle corresponds to a cycle of disks bounded by these curves and homeomorphisms between them, up to isotopy. In the notation of [2], this corresponds to a *degenerate* Levy cycle.

Proof of the main theorem. Recall the hypothesis that all critical orbits are periodic. As each $\gamma_i \in \Gamma$ will enclose at least one critical point in its cycle, the degree of the mapping cannot be one, so that there can be no Levy cycle. □

The proof strategy above in fact works even under the weaker hypothesis that all cycles in P_f contain a critical point.

4 Final Remarks

After having finished the arguments above, the author recognized that most of the proof strategy is subsumed as a particular case of Theorem 3.1 of [2], in particular item (4).

References

[1] A. Douady and J. Hubbard, *A proof of Thurston's topological characterization of rational functions*, Acta Math. **171** (1993), 263-297.

[2] T. Lei, *Matings of quadratic polynomials*, Ergodic Theory & Dynamical Systems **12** (1992), 589–620.