Quasiconformal Maps

Eduardo Ventilari Sodré

Department of Mathematics, Brown University

Fall 2023

1 Introduction

Conformal maps in complex analysis enjoy many nice geometrical properties. Apart from preserving angles, they posses smoothness, removability of singularities and stability under locally uniform convergence, to name a few well known results.

However, at times, a more general class of maps is needed. This is so in many different contexts: when studying certain objects in complex dynamics, understanding the difference between hyperbolic structures on Riemann surfaces, and, perhaps more historically accurate, understanding which maps extremize some mapping problem. This paves the way for the notion of quasiconformal maps, which intuitively are maps that distort the conformal geometry of objects in a bounded way. They have properties analogous (and sometimes identical) to conformal maps, making their study particularly rich. There are two common definitions of quasiconformal maps, one of geometric nature and the other analytic. Both are important for understanding them, and the interplay between the geometry and analysis is crucial for their applications.

2 The Geometric Definition

2.1 The Modulus of a Quadrilateral

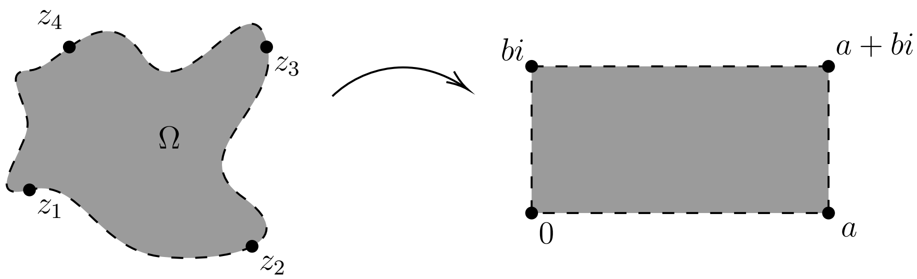
Quasiconformal maps may be first understood by certain geometrical invariants that conformal maps preserve, and considering which maps generalize this invariance in a controlled form.

Consider a Jordan domain $\Omega \subset \hat{\mathbb{C}}$, that is, a connected open set Ω such that $\partial\Omega$ is a simple closed curve. It is a theorem that if $f : \Omega \to \Omega'$ is a conformal map between two Jordan domains, then f extends continuously to the boundary as an orientation preserving homeomorphism $f : \overline{\Omega} \to \overline{\Omega'}$. Such a map is uniquely determined by the image of three distinct boundary points of $\partial\Omega$, because any automorphism of $\mathbb{D}$ is given by a Möbius transformation, which is determined by the the image of three points on $\partial\mathbb{D}$.

Given distinct points $z_1, z_2, z_3, z_4 \in \partial\Omega$ in cyclic order, we say that Ω is a **quadrilateral** $Q(z_1, z_2, z_3, z_4)$. By the Riemann mapping theorem, we may find a conformal map from Ω to the upper half-plane $\mathbb{H}$, and by a suitable elliptic integral, a conformal map from Ω to the rectangle

$$R = (0, a) \times (0, b) \subset \mathbb{C}$$

mapping the vertices to 0, a, $a + bi$ and bi in cyclic order. Call the arcs (z_1, z_2) and (z_3, z_4) to be the a-sides of Q, and likewise (z_2, z_3), (z_4, z_1) to be the b-sides.



The value $a/b > 0$ is unique:

Proposition 2.1. *If $f : R \to R'$ is a conformal map mapping vertices to vertices in cyclic order and sides to sides, then $a/b = a'/b'$.*

Proof. By the Schwarz reflection principle, we may extend the map to the whole complex plane by reflecting about each of side of the rectangle sucessively, "tiling" $\mathbb{C}$ by reflected copies of R. The resulting map is a conformal entire map, hence affine; since it fixes the origin and maps a to a', it must be the map $a'z/a$. Hence $a'b/a = b'$, so $a/b = a'/b'$. □

This allows us to define the **modulus** $M(Q(z_1, z_2, z_3, z_4))$ of the quadrilateral as the ratio $a/b \in (0, +\infty)$. As a consequence, the modulus of a quadrilateral is a conformal invariant. Note that

$$M(Q(z_2, z_3, z_4, z_1)) = M(Q(z_1, z_2, z_3, z_4))^{-1}.$$

We search for a characterization of the modulus that does not rely on conformal maps. Consider again $f : Q \to R$ a conformal map from Q onto a rectangle R of length a and height b, where $M(Q) = a/b$. The area of R is given by

$$\iint_Q |f'(z)|^2 dxdy = ab.$$

Let Γ be the family of locally rectifiable arcs joining the sides (z_1, z_2) and (z_3, z_4) of Q; for all $\gamma \in \Gamma$,

$$\int_\gamma |f'(z)||dz| \geq b$$

with equality if and only if γ is the preimage of a vertical line in R. Hence

$$M(Q) = \frac{\iint_Q |f'(z)|^2 dxdy}{\left(\inf_{\gamma\in\Gamma} \int_\gamma |f'(z)||dz|\right)^2}.$$

This length-area ratio is generalized is the following way. Let P be the family of non-negative, measurable and locally integrable functions ρ on Q such that $A_\rho(Q) := \iint_Q \rho^2 dxdy \neq 0, \infty$. Intuitively, P is the family of conformal metrics on Q, and we may calculate the length of a curve $\gamma \in \Gamma$ by $L_\rho(\gamma) := \int_\gamma \rho(z)|dz|$.

Proposition 2.2.

$$M(Q) = \inf_{\rho\in P} \frac{A_\rho(Q)}{(\inf_{\gamma\in\Gamma} L_\rho(\gamma))^2},$$

where the infimum is attained by $\rho(z) = |f'(z)|$, f *being the canonical conformal map taking* Q *to the rectangle* $(0, a) \times (0, b)$.

Proof. Given $\rho \in P$, we obtain a corresponding function $\tilde{\rho}(z) = \rho(z)/|f'(z)|$. Then

$$A_\rho(Q) = \iint_Q \rho^2 dxdy = \iint_Q \tilde{\rho}^2 |f'(z)|^2 dxdy = \iint_R \tilde{\rho}^2 dudv,$$

and

$$L_\rho(\gamma) = \int_\gamma \rho|dz| = \int_\gamma \tilde{\rho}|f'(z)||dz| = \int_{f\circ\gamma} \tilde{\rho}|dw|.$$

Letting $l = \inf_{\gamma\in\Gamma} L_\rho(\gamma)$, for all $u \in [0, a]$ we have that

$$l \leq \int_0^b \tilde{\rho}(u + iv)dv \implies al \leq \int_0^a \int_0^b \tilde{\rho}dudv,$$

and by Fubini and Cauchy-Schwarz, we have

$$(al)^2 \leq \left(\iint_R \tilde{\rho} dudv\right)^2 \leq \left(\iint_R \tilde{\rho}^2 dudv\right)\left(\iint_R 1 dudv\right) = A_\rho(Q)ab,$$

which then shows the inequality. □

By considering $\rho \equiv 1$ the usual euclidian metric on Q, with area $A(Q)$ and length of curves $L(\gamma)$, define the value $s_a := \inf_{\gamma\in\Gamma} L(\gamma)$ to be the minimum distance between the a-sides of the quadrilateral with respect to the euclidean metric, and likewise s_b to be the minimum distance between the b-sides. We obtain the crucial inequality:

Corollary 2.3 (Rengel's Inequality)**.**

$$\frac{s_b^2}{A(Q)} \leq M(Q) \leq \frac{A(Q)}{s_a^2},$$

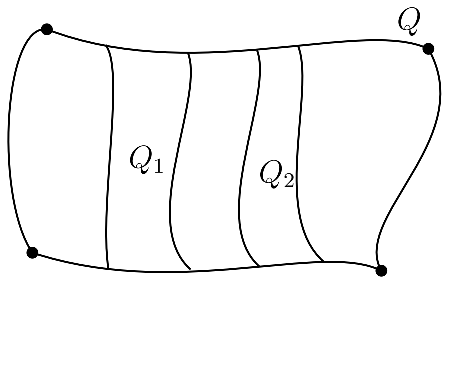
with equality in both cases if and only if Q is a rectangle.

Equality happens exactly when $|f'(z)| \equiv 1$, so that f is a rotation. Rengel's inequality has several nice consequences:

Proposition 2.4 (Superadditivity of the modulus)**.** *Given quadrilaterals Q_n for $n \in \mathbb{N}$ with disjoint interiors and a quadrilateral Q such that $\overline{Q_n} \subseteq \overline{Q}$ for all n, and such that the a-sides of each Q_n are contained one in each a-side of Q, we have that*

$$\sum_{n=1}^{\infty} M(Q_n) \leq M(Q).$$

If Q is a rectangle, then equality holds if and only if each Q_n is a rectangle and $\sum_{n=1}^{\infty} A(Q_n) = A(Q)$.



Proof. We may assume Q is a rectangle of height 1. As $s_a(Q_n) \geq 1$, we have by Rengel's inequality that $M(Q_n) \leq \frac{A(Q_n)}{s_a^2} \leq A(Q_n)$, so

$$\sum M(Q_n) \leq \sum A(Q_n) \leq A(Q) = M(Q),$$

with equality holding if and only if each Q_n is a rectangle and the sum of the areas equals the area of Q. $\square$

Suppose Q is a quadrilateral with sides a_1, b_1, a_2, b_2. We say that a sequence of quadrilaterals Q_n with sides $a_1^n, b_1^n, a_2^n, b_2^n$ *converges from the inside* to Q if $\overline{Q_n} \subset Q$ if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for $n \geq N_\varepsilon$, the sides a_i^n, b_i^n lie within a strip of width ε of the sides a_i, b_i respectively.

Proposition 2.5 (Continuity of the modulus from inside)**.** *Suppose Q_n converges from the inside to Q. Then $M(Q_n) \to M(Q)$.*

Proof. We may assume Q is a rectangle $R = [0, M(Q)] \times [0, 1]$ by a suitable conformal map $f : Q \to R$, which is uniformly continuous on $\overline{Q}$. The images Q'_n of the quadrilaterals Q_n by the conformal map converge from the inside to R, so that we have $s_b'^n \to M(Q)$ and $s_a'^n \to 1$; since we also have $A(Q'_n) \leq A(R) = M(Q)$, taking the limit in Rengel's inequality

$$\frac{(s_b'^n)^2}{M(Q)} \leq \frac{(s_b'^n)^2}{A(Q'_n)} \leq M(Q'_n) \leq \frac{A(Q'_n)}{(s_a'^n)^2} \leq \frac{M(Q)}{(s_a'^n)^2}$$

gives us that $M(Q_n) = M(Q'_n) \to M(R) = M(Q)$. $\square$

A more general continuity property of the modulus is true, not just for quadrilaterals converging from the inside, but the above is sufficient for our purposes. It is also worth noting that for a given quadrilateral Q, we may always construct a sequence Q_n of quadrilaterals converging to Q from the inside and having analytic arcs. This can be seen by mapping Q to the unit disk with four marked points on the boundary, and taking Q'_n to be disks of radius $1 - 1/n$, with corresponding boundary points radially aligned.

2.2 Quasiconformal Maps

Given an open set $U \subseteq \hat{\mathbb{C}}$ and $K \in [1, \infty)$, we say that $f : U \to \hat{\mathbb{C}}$ is a **K-quasiconformal map** if f is an orientation preserving homeomorphism onto its image and, for all quadrilaterals $\overline{Q} \subset U$, we have that

$$M(f(Q)) \leq KM(Q).$$

Note that, given a quadrilateral $Q(z_1, z_2, z_3, z_4)$ and taking $Q(z_4, z_1, z_2, z_3)$, if f is K-quasiconformal we in fact obtain the double inequality

$$\frac{1}{K} M(Q) \leq M(f(Q)) \leq K M(Q)$$

for all quadrilaterals $\overline{Q} \subset U$. We define the **dilation** of f as

$$K(f) := \sup_{\overline{Q} \subset U} \frac{M(f(Q))}{M(Q)}.$$

It is evident from the definition the following properties:

i. A conformal map is 1-quasiconformal;

ii. The inverse of a K-quasiconformal map is K-quasiconformal;

iii. if f is K_1-quasiconformal and g is K_2-quasiconformal, then $g \circ f$ is $K_1 K_2$-quasiconformal.

We may recover conformality from 1-quasiconformality:

Theorem 2.6. *A 1-quasiconformal map is conformal.*

Proof. Given a 1-quasiconformal map $f : U \to V$, it is sufficient to show conformality in a quadrilateral $\overline{Q} \subset U$. We map Q and $f(Q)$ to rectangles by conformal maps, which preserve the modulus. Hence we may consider the situation of a 1-quasiconformal map from $[0, M] \times [0, 1]$ to $[0, M] \times [0, 1]$. Consider $z = (x, y) \in Q$. By dividing the rectangle into two rectangles $R_1 = [0, x] \times [0, 1]$ and $R_2 = [x, M] \times [0, 1]$, we see that $M(R_1) + M(R_2) = M$. As $M(f(R_i)) = M(R_i)$, we have that $f(R_1), f(R_2)$ satisfy the equality in 2.4, so that $f(R_1)$ and $f(R_2)$ are rectangles. Since $x = M(R_1) = M(f(R_1))$, we have that the real part of $f(z)$ is x. By repeating this argument with respect to y, we conclude that $f(z) = z$, and f is the identity. □

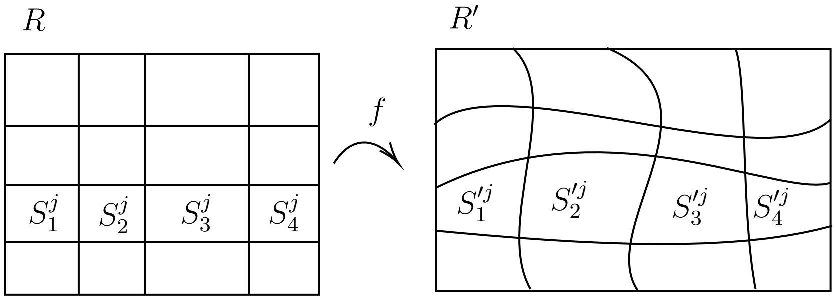
By continuity of the modulus from inside, we have the following useful fact:

Lemma 2.7. *Let Q be a quadrilateral and $f : \overline{Q} \to \overline{Q'}$ an orientation preserving homeomorphism to a quadrilateral Q' which is K-quasiconformal in its interior. Then $M(Q') \leq K M(Q)$.*

This next result shows that quasiconformality is a local condition:

Theorem 2.8. *If $f : U \to V$ is a orientation preserving homeomorphism such that for all $z \in U$, f is K-quasiconformal in a neighborhood of z, then f is K-quasiconformal.*

Proof. It is sufficient to consider the case of a homeomorphism between rectangles $f : \overline{R} \to \overline{R'}$, where $R = [0, M] \times [0, 1]$ and $R' = [0, M'] \times [0, 1]$, and show that $M(R') \leq KM(R)$. By a compactness argument, we may find a grid on R where on each rectangle of the grid, f is K-quasiconformal. Consider horizontal lines connecting the b-sides of R with distance less than h apart, including the horizontal lines of the grid. As f is uniformly continuous, we may take h such that, for all horizontal strips $S^j = [0, M] \times [y_j, y_{j+1}]$ and the smaller substrips S_i^j which the vertical lines of the grid divide S^j into, the images of the b-sides of each S_i^j have the sum of their lengths smaller than some given $\varepsilon > 0$.



Let S'^j, $S_i'^j$ be the images of these strips. By our choice of h, we have

$$\sum_i s_b(S_i'^j) > M' - \varepsilon,$$

and by Rengel's inequality, $M(S_i'^j) \geq \frac{s_b(S_i'^j)^2}{A(S_i'^j)}$. Hence by Cauchy-Schwarz,

$$\sum_i M(S_i'^j) \geq \sum_i \frac{s_b(S_i'^j)^2}{A(S_i'^j)} \geq \frac{\left(\sum_i s_b(S_i'^j)\right)^2}{\sum_i A(S_i'^j)} > \frac{(M' - \varepsilon)^2}{A(S'^j)}.$$

By our hypotheses and lemma 2.7, $M(S_i'^j) \leq KM(S_i^j)$, and

$$\sum_i M(S_i'^j) \leq KM(S^j) = \frac{KM}{y_{j+1} - y_j},$$

so that

$$\frac{y_{j+1} - y_j}{KM} \leq \frac{A(S'^j)}{(M' - \varepsilon)^2},$$

and by summing over all horizontal strips, we have $\frac{1}{KM} \leq \frac{M'}{(M'-\varepsilon)^2}$. Taking the limit as $\varepsilon \to 0$ implies the result. $\square$

There are two other notable ways of characterizing quasiconformal maps by conformal invariants. The first is by the modulus of ring domains. Any doubly connected domain B of $\hat{\mathbb{C}}$ (that is, such that $\hat{\mathbb{C}} \setminus B$ has exactly two connected components) can be conformally mapped onto an annulus

$$A_{r,R} = \{z\colon r < |z| < R\},$$

where $0 \leq r < R \leq \infty$. The number $M(B) = \frac{1}{2\pi}\log(R/r)$ is a conformal invariant of B, defining the modulus of a ring domain, with analogous properties to the modulus of a quadrilateral. In fact, we can prove that an orientation preserving homeomorphism $f : U \to V$ is K-quasiconformal if and only if for every ring domain R such that $\overline{R} \subset U$, we have $M(f(R)) \leq KM(R)$ [5].

The second is by the extremal length of path families Γ in a domain. It is defined as

$$\mathcal{L}(\Gamma) \coloneqq \sup_{\rho \in P} \frac{(\inf_{\gamma\in\Gamma} L_\rho(\gamma))^2}{A(\rho)},$$

where again P is the family of conformal metrics in the domain and $A(\rho)$ is the area. This notion subsumes the previous definitions of the module of quadrilaterals and of ring domains, and likewise it is a theorem that an orientation preserving homeomorphism f is K-quasiconformal if and only if for all path families, $\mathcal{L}(f(\Gamma)) \leq K\mathcal{L}(\Gamma)$.

3 The Analytic Definition

3.1 $\mathbb{R}$-linear Maps

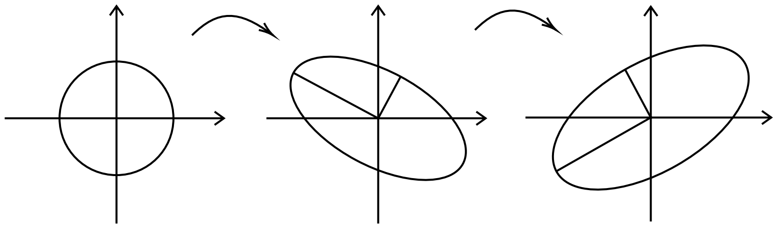
Any $\mathbb{R}$-linear orientation-preserving map $L : \mathbb{C} \to \mathbb{C}$ can be uniquely written in the form

$$L(z) = Az + B\overline{z}, \quad A, B \in \mathbb{C},$$

where $\det L = |A|^2 - |B|^2 > 0$. Note that L is $\mathbb{C}$-linear if and only if $B = 0$. Letting $A = |A|e^{i\alpha}$, we may also write

$$L(z) = e^{i\alpha}|A|(z + \frac{B}{A}\overline{z}),$$

which is a composition of a rotation, a dilation and a self-adjoint map $z \mapsto z + \frac{B}{A}\overline{z}$. The latter has two orthogonal eigenvectors, and this illustrates that geometrically L maps circles to ellipses.



The **Beltrami coefficient** of L is defined as

$$\mu(L) := \frac{B}{A} = \left|\frac{B}{A}\right| e^{i2\theta} \in \mathbb{D},$$

and the **dilation** $\mathrm{Dil}(L)$ is the real value

$$\mathrm{Dil}(L) := \frac{1+|\mu|}{1-|\mu|} = \frac{|A|+|B|}{|A|-|B|} \in [1,\infty).$$

If $\|L\|$ denotes the operator norm of L, we may recover $\mathrm{Dil}(L)$ also as

$$\mathrm{Dil}(L) = \frac{\|L\|^2}{\det L},$$

where $\|L\| = \sup_{\|u\|\leq 1} \|Lu\| = |A| + |B|$ is the operator norm of L.

Geometrically, if $E(L)$ denotes the ellipse which is the inverse image of the unit circle by L, then $\mathrm{Dil}(L)$ is the ratio of the major and minor axes of $E(L)$, which are in the directions $e^{i(\theta+\pi/2)}$ and $e^{i\theta}$ respectively. As a $\mathbb{C}$-linear map maps circles to circles, intuitively the dilation of a an $\mathbb{R}$-linear map measures its deviation to a $\mathbb{C}$-linear map, expressing the ratio of stretching in orthogonal directions. A standard example is the map $(x,y) \mapsto ((b/a)x, y)$ for $0 < a \leq b$, mapping a rectangle of sidelengths $a \times 1$ diffeomorphically onto a rectangle of sidelengths $b \times 1$, preserving the sides; in this case the dilation is b/a.

3.2 Quasiconformal Diffeomorphisms

Let U and V be domains in $\mathbb{C}$ and $f : U \to V$ a C^1 orientation-preserving diffeomorphism, where we write

$$f(z) = f(x+iy) = u(x+iy) + iv(x+iy),$$

and u, v can be seen as $\mathbb{R}$-valued functions in $\mathbb{R}^2$. Recall that f is conformal if and only if it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

which is equivalent to the differential $df_z : \mathbb{C} \to \mathbb{C}$ being $\mathbb{C}$-linear, that is,

$$\begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We define the **Wirtinger derivatives** by the formulas

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right);$$

conformality of f is equivalent to $\partial_{\overline{z}} f = 0$. In general, the differential of f may be written as

$$df_z(w) = \partial_z f(w) w + \partial_{\overline{z}} f(w)\overline{w},$$

or simply

$$df_z = \partial_z f dz + \partial_{\overline{z}} f d\overline{z}.$$

Note that as f is orientation preserving, $|\partial_{\overline{z}} f| < |\partial_z f|$ for all $z \in U$. We can talk about the dilation of df_z for each $z \in U$, obtaining the Beltrami coefficient $\mu_f : U \to \mathbb{D}$ given by $\partial_{\overline{z}} f / \partial_z f$, and the dilation at each point

$$\mathrm{Dil}_z(f) = \frac{1 + |\mu_f|}{1 - |\mu_f|} = \frac{|\partial_z f| + |\partial_{\overline{z}} f|}{|\partial_z f| - |\partial_{\overline{z}} f|} = \frac{\|df_z\|^2}{\mathrm{Jac}_z(f)},$$

where $\mathrm{Jac}_z(f) = \det(df_z) > 0$ is the determinant of the Jacobian of f at z.

With this notation, given $K \geq 1$, we say that $f : U \to V$ is a K-**quasiconformal diffeomorphism** if

$$\sup_{z \in U} \mathrm{Dil}_z(f) \leq K.$$

In this case we denote $\mathrm{Dil}(f) := \sup_{z \in U} \mathrm{Dil}_z(f)$. Equivalently, for $k = \dfrac{K-1}{K+1}$, where $0 \leq k < 1$, f is quasiconformal if for all $z \in U$ we have

$$|\partial_{\overline{z}} f| \leq k |\partial_z f|.$$

This essentially says that the infinitemisal ratio of stretching of f is bounded; observe that a 1-quasiconformal diffeomorphism is just a conformal map. From the usual chain rule of derivatives applied to ∂_z and $\partial_{\overline{z}}$, we may also deduce the following identity for the Beltrami coefficient of a composition $g \circ f$ of quasiconformal diffeomorphisms:

$$\mu_g \circ f = \frac{\partial_z f}{\partial_{\overline{z}} \overline{f}} \frac{\mu_{g \circ f} - \mu_f}{1 - \overline{\mu_f} \mu_{g \circ f}}. \tag{1}$$

In particular, composing quasiconformal diffeomorphisms with conformal maps either on the left or on the right preserves the dilation.

Since there is no conformal map between rectangles of different modules, the following inequality indicates which C^1 orientation preserving diffeomorphism between them minimizes the dilation:

Theorem 3.1 (Grötzsch)**.** *Let $R = [0,a] \times [0,b]$, $R' = [0,a'] \times [0,b']$ and $f : R \to R'$ be a C^1 orientation preserving diffeomorphism mapping a-sides to a'-sides and b-sides to b'-sides, where we assume $a'/b' \geq a/b$. Then*

$$\mathrm{Dil}(f) \geq \frac{a'/b'}{a/b},$$

with equality if and only if f is the affine map $(x,y) \mapsto ((b'/b)x, (a'/a)y)$.

Proof. Consider a horizontal line in R, so that its image, parametrized by $f(x+iy)$ for a fixed y, will have length not less than a':

$$a' \leq \int_0^a |\partial_x f(x+iy)| dx \leq \int_0^a \|df_z(x+iy)\| dx.$$

Since this is true for all y, we have

$$a'b \leq \int_0^b \int_0^a \|df_z\| dxdy = \iint_R \sqrt{\mathrm{Jac}_z(f)} \frac{\|df_z\|}{\sqrt{\mathrm{Jac}_z(f)}} dxdy.$$

Applying Cauchy-Schwarz,

$$\begin{aligned}(a'b)^2 &\leq \iint \mathrm{Jac}_z(f) dxdy \iint_R \frac{\|df\|^2}{\mathrm{Jac}_z(f)} dxdy = a'b' \iint_R \mathrm{Dil}_z(f) dxdy \\ &\leq a'b' \iint_R \mathrm{Dil}(f) dxdy = (a'b')(ab)\,\mathrm{Dil}(f),\end{aligned}$$

thereby proving the inequality. Equality happens only when $\mathrm{Dil}_z(f)$ is a.e. equal to $(a'/b')/(a/b)$, $|\partial_x f|$ is a.e. equal to $\|df_z\|$, and $\mathrm{Dil}_z(f)$ is a.e. equal to $c\|df_z\|$, where c is some constant. From

$$a'b = \iint_R \|df_z\| dxdy,$$

as $\|df_z\|$ is a.e. constant, this will imply that $\|df_z\|$ is a.e. equal to a'/a, and hence $\mathrm{Jac}_z(f)$ is a.e. equal to $(a'/a)(b'/b)$. This implies that a.e. df_z is of the form

$$\begin{bmatrix} a'/a & 0 \\ 0 & b'/b \end{bmatrix},$$

and by integrating, f is the aformentioned affine map. $\square$

The preceding proof in fact shows that the affine map has both least maximal dilation and least average dilation among all C^1 orientation preserving diffeomorphisms. We obtain the following crucial corollary:

Corollary 3.2. *A K-quasiconformal diffeomorphism is K-quasiconformal in the geometric sense.*

Proof. For a quadrilateral $\overline{Q} \subset U$, we may compose f with suitable conformal maps so that the problem is reduced to theorem 3.1, and

$$\frac{M(f(Q))}{M(Q)} \leq \mathrm{Dil}(f).$$

$\square$

As a partial converse, we have:

Theorem 3.3. *If a K-quasiconformal map is differentiable at a point z_0, then $\mathrm{Dil}_{z_0}(f) \leq K$.*

Proof. By composing f with translations and rotations, we may assume that $z_0 = f(z_0) = 0$ and df_z is of the form $z \mapsto Az + B\overline{z}$, for $A, B > 0$. Then

$$f(z) = Az + B\overline{z} + o(z).$$

We consider $\varepsilon > 0$ small so that the square $S_\varepsilon = [-\varepsilon, \varepsilon]^2$ is in the domain of f. It follows that

$$s_b(f(S_\varepsilon)) = 2\varepsilon(A + B) + o(\varepsilon),$$

and that $f(S_\varepsilon)$ has area

$$A(f(S_\varepsilon)) = 4\varepsilon^2(A^2 - B^2) + o(\varepsilon^2).$$

By Rengel's inequality and that $M(S_\varepsilon) = 1$, we have

$$\frac{4\varepsilon^2(A + B)^2 + o(\varepsilon^2)}{4\varepsilon^2(A^2 - B^2) + o(\varepsilon^2)} \leq M(f(S_\varepsilon)) = \frac{M(f(S_\varepsilon))}{M(S_\varepsilon)} \leq K,$$

so that

$$(A + B)^2 + o(\varepsilon^2)/\varepsilon^2 \leq K((A^2 - B^2) + o(\varepsilon^2)/\varepsilon^2).$$

by taking $\varepsilon \to 0$, we obtain that $\mathrm{Dil}_{z_0}(f) \leq K$. $\square$

These two results combine to show that a C^1 orientation preserving diffeomorphism is a K-quasiconformal map if and only if $\mathrm{Dil}(f) \leq K$.

3.3 Quasiconformal Maps

Quasiconformal maps are not in general diffeomorphisms, but it was discovered that they possess a great deal of smoothness in a suitably general context. There are two commonly used equivalent analytic definitions for quasiconformal maps, generalizing the differential inequality for quasiconformal diffeomorphisms. One hightlights the concept of absolute continuity on lines (ACL), and the other utilizes distributional derivatives.

A function $f : [a,b] \to \mathbb{C}$ defined on a compact interval is said to be **absolutely continuous** if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $(a_k, b_k) \subset [a,b]$ are finitely many disjoint intervals and $\sum_k (b_k - a_k) < \delta$, then $\sum_k |f(b_k) - f(a_k)| < \varepsilon$. This is the most general class of functions for which the fundamental theorem of calculus is true: f is absolutely continuous if and only if $f'(t)$ exists almost everywhere, is integrable and $f(x) - f(a) = \int_a^x f'(t)dt$ [7].

A function $f : U \subset \mathbb{C} \to \mathbb{C}$ is **absolutely continuous on lines (ACL)** if, for every rectangle R, f is absolutely continuous on almost all vertical lines of R and on almost all horizontal lines of R. As a consequence, if f is ACL, it has partial derivatives almost everywhere. The following lemma, whose proof in [1] uses Egorov's theorem and points of Lebesgue density, is relevant for what follows.

Lemma 3.4. *If f is a homeomorphism and has partial derivatives almost everywhere, then f is differentiable almost everywhere. Moreover, the Jacobian* $\mathrm{Jac}(f)$ *is locally integrable.*

Given $k = (K-1)/(K+1)$, we say that an orientation preserving homeomorphism $f : U \to V$ is **K-quasiconformal** if it is ACL and, for almost every $z \in U$,

$$|\partial_{\overline{z}} f| \leq k |\partial_z f|.$$

Since

$$\mathrm{Dil}_z(f) = \frac{(|\partial_z f| + |\partial_{\overline{z}} f|)^2}{\mathrm{Jac}_z(f)} \leq K \implies |\partial_{\overline{z}} f|^2 \leq |\partial_z f|^2 \leq K \,\mathrm{Jac}_z(f),$$

we obtain that the partial derivatives are in fact locally L^2-integrable. We also see that, as a consequence of being ACL, $\partial_z f$ and $\partial_z f$ will be weak derivatives of f in the distributional sense: for any test function $\varphi \in C_c^\infty(U)$,

$$\iint_U (\partial_z f)\varphi = -\iint_U f \partial_z \varphi, \qquad \iint_U (\partial_{\overline{z}} f)\varphi = -\iint_U f \partial_{\overline{z}} \varphi.$$

Conversely [1]:

Proposition 3.5. *If f has locally integrable distributional derivatives, then f is ACL.*

This shows that an equivalent analytic definition for quasiconformal maps is as such. Given $K \geq 1$ and $k = (K-1)/(K+1)$, a map $f : U \to V$ is K**-quasiconformal** if it is an orientation preserving homeomorphism, it has locally L^2-integrable distributional derivatives and, for almost every $z \in U$,

$$|\partial_{\bar{z}} f| \leq k |\partial_z f|.$$

It is a remarkable fact of the theory of quasiconformal maps that the geometric and analytic definitions coincide.

Proposition 3.6. *A K-quasiconformal map in the analytic sense is K-quasiconformal in the geometric sense.*

The proof of this is essentially the same as in theorem 3.1, where all notions carry over for the absolute continuity on lines and local integrability of the Jacobian. The only nuance understanding what happens to the distributional partial derivatives under composition with conformal maps, whereby the usual chain rule of derivatives applies almost everywhere.

Proposition 3.7. *A K-quasiconformal map in the geometric sense is K-quasiconformal in the analytic sense.*

Proof. First we prove that f is ACL; given a rectangle $R = [a,b] \times [c,d] \subset U$, and for $y \in [c,d]$, let $A(y) = m(f([a,b] \times [c,y]))$. It is a bounded incresing function, so that its derivative exists almost everywhere [7]. We show that f is absolutely continuous on $[a,b] \times \{y\}$ if $A'(y)$ exists; by an analogous argument for vertical lines, f will be ACL.

If $(a_1, b_1), \ldots, (a_k, b_k)$ are finite disjoint intervals in $[a,b]$, consider the rectangles $R_j = [a_j, b_j] \times [y, y+h]$ for some small $h > 0$, and α_j and β_j the points (a_j, y) and (b_j, y) respectively. Note that $|f(\beta_j) - f(\alpha_j)| \leq s_b(f(R_j))$, and by Rengel's inequality,

$$\frac{|f(\beta_j) - f(\alpha_j)|^2}{m(f(R_j))} \leq M(f(R_j)) \leq KM(R_j) = K\frac{b_j - a_j}{h}.$$

Applying Cauchy-Schwarz with the fact that $\sum m(f(R_j)) \leq A(y)$, we obtain

$$\begin{aligned}\left(\sum_{j=1}^{k} |f(\beta_j) - f(\alpha_j)|\right)^2 &\leq \left(\sum_{j=1}^{k} \frac{|f(\beta_j) - f(\alpha_j)|^2}{m(f(R_j))}\right)\left(\sum_{j=1}^{k} m(f(R_j))\right) \\ &\leq \frac{K}{h}\sum_{j=1}^{k}(b_j - a_j)A(y),\end{aligned}$$

so that, as $h \to 0$, we have

$$\left(\sum_{j=1}^{k} |f(\beta_j) - f(\alpha_j)|\right)^2 \leq KA'(y) \sum_{j=1}^{k} (b_j - a_j),$$

showing absolute continuity of f in $[a, b] \times \{y\}$.

Since f is ACL, it is differentiable almost everywhere. By theorem 3.3, $\mathrm{Dil}_z(f) \leq K$ wherever f is differentiable, which concludes that f is analytically K-quasiconformal. $\square$

With the analyitic definition for quasiconformal maps, the Beltrami coefficient $\mu_f(z)$ is a well defined function in $L^\infty(U)$, where $\|\mu_f\|_\infty \leq k < 1$, and f is a weak solution of the differential equation

$$\partial_{\overline{z}} f(z) = \mu(z) \partial_z f(z).$$

Given any measurable function $\mu : U \to \mathbb{C}$ such that $\|\mu\|_\infty < 1$, we call

$$\partial_{\overline{z}} f = \mu \partial_z f \tag{2}$$

a **Beltrami equation**. It is straightfoward to see, from what we have deduced previously, that a homeomorphism is quasiconformal if and only if it is a solution of a Beltrami equation, in the sense that f is ACL, has locally L^2-integrable derivatives and they satisfy the equation almost everywhere. Or equivalently, f has locally L^2 distributional derivatives and they satisfy (2). Rephrasing the condition of quasiconformality in terms of solutions of a partial differential equation allows new insights into it. One of the main ones is the measurable Riemann mapping theorem, which asserts that such an equation always has solution in $\mathbb{C}$, unique up to a specific normalization condition. A more in depth discussion would take us too far into the beautiful singular integral operators, such as the Hilbert and Ahlfors-Beurling transforms: a treatment may be found in [5].

References

[1] L. Ahlfors, *Lectures on Quasiconformal Maps*, D. Van Nostrand Company, 1966.

[2] B. Branner and N. Fagella, *Quasiconformal Surgery in Holomorphic Dynamics*, Cambridge University Press, 2014.

[3] J. Hubbard, *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics, Volume 1: Teichmüller Theory*, Matrix Editions, 2006.

[4] J. Kahn and M. Fortier Bourque, *Univalent Functions and Teichmüller Spaces*, 2014.

[5] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag New York, 1987.

[6] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag New York Heidelberg Berlin, 1973.

[7] E. Stein and R. Shakarchi, *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*, Princeton University Press, 2005.