Lower Bound for Smallest Geodesic

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Let $f : S^2 \to S^2$ be a postcritically finite branched covering map, Ω_f the critical set, $P_f = \bigcup_{n>0} f^n(\Omega_f)$ the postcritical set, and $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$ be the Thurston pullback map on the Teichmuller space of (S^2, P_f). If $\tau \in \mathcal{T}_f$ is represented by a diffeomorphism $\varphi : (S^2, P_f) \to \mathbb{P}^1$, then $\sigma_f(\tau)$ is represented by the diffeomorphism $\phi' : (S^2, P_f) \to \mathbb{P}^1$ such that f_τ in the following commutative diagram is a holomorphic map:

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\phi'} & \mathbb{P}^1 \\ f \downarrow & & \downarrow f_\tau \\ (S^2, P_f) & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

We suppose that f has no obstruction, which means that, when the corresponding orbifold is hyperbolic, $\tau_i \coloneqq \sigma_f^i(\tau)$ converges to a unique fixed point in $\mathcal{T}_f$, or equivalently, for every f-stable multicurve Γ in (S^2, P_f), the pullback map $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$, defined by

$$f_\Gamma(\gamma_j) = \sum_{i,\alpha} \frac{1}{d_{i,j,\alpha}} \gamma_i,$$

has biggest eigenvalue $\lambda(f, \Gamma) < 1$. Since there are only finitely many possible matrices f_Γ for an f-stable multicurve Γ, this implies that there exists a uniform bound $\lambda(f, \Gamma) \leq \lambda < 1$ for all Γ. Moreover, there exists a smallest $m > 0$ such that $\|f_\Gamma^m\| < \frac{1}{2}$. This is may be deduced as a consequence of the von Neumann formula for the spectral radius and the fact that there are finitely many matrices of the form f_Γ.

Given an essential closed curve γ in (S^2, P_f), let $l_\tau(\gamma)$ be the length of the correspoding closed geodesic in τ in the homotopy class of $\phi(\gamma)$. Also define

$$\omega(\tau, \gamma) \coloneqq -\log l_\tau(\gamma).$$

Given an initial point $\tau_0 \in \mathcal{T}_f$ and $D \coloneqq d(\tau_0, \tau_1)$, we want to find a lower bound for the length of the shortest closed geodesic of τ_i along the iteration by the Thurston pullback map. This is equivalent to finding an upper bound for

$$\omega(\tau) \coloneqq \sup_{\gamma} \omega(\tau, \gamma).$$

Since $(f_\Gamma)^m = (f^m)_\Gamma$, let $\phi(P_f) = P$, $\phi'(P_f) = P'$, and $(f_\tau^m)^{-1}(P) = P''$, so that $f_\tau^m : \mathbb{P}^1 \setminus P'' \to \mathbb{P}^1 \setminus P$ is an unbranched holomorphic covering and $\mathbb{P}^1 \setminus P'' \hookrightarrow \mathbb{P}^1 \setminus P'$ is a holomorphic injection. This will allow us to compare the "lengths" $\omega(\tau, \gamma)$, $\omega(\tau, \gamma')$ and $\omega(\tau', \gamma)$, for an essential closed curve γ in (S^2, P_f) and γ' a component of its preimage, where $\tau' = \sigma_f^m(\tau)$.

Recall the following:

Lemma 0.1. *The functions $\tau \mapsto \omega(\tau, \gamma)$ and $\tau \mapsto \omega(\tau)$ are Lipschitz, with Lipschitz constant* 2.

As $f_\tau^m : \mathbb{P}^1 \setminus P'' \to \mathbb{P}^1 \setminus P$ is a holomorphic covering map, if γ' is a component of $(f^m)^{-1}(\gamma)$, then

$$l_{\mathbb{P}^1 \setminus P''}(\gamma') = d_\alpha l_\tau(\gamma) \leq d^m l_\tau(\gamma).$$

Moreover, because of the analytic inclusion $\mathbb{P}^1 \setminus P'' \hookrightarrow \mathbb{P}^1 \setminus P'$, we also have $l_{\mathbb{P}^1 \setminus P''}(\gamma') \geq l_{\tau'}(\gamma')$, so that

$$\omega(\tau', \gamma') \geq \omega(\tau, \gamma) - m \log d.$$

If τ and τ' are points in the orbit of τ_0, we also have that $d(\tau, \tau') \leq mD$, so that, by the Lipschitz estimate,

$$\omega(\tau, \gamma) - m \log d \leq \omega(\tau', \gamma') \leq \omega(\tau, \gamma') + 2mD,$$

hence

$$\omega(\tau, \gamma) \leq \omega(\tau, \gamma') + m(\log d + 2D).$$

In order to compare the growth of $\omega(\tau, \gamma)$ with the map f_Γ, we must find a way to guarantee that if γ is (in the homotopy class of) the shortest geodesic of τ, there is an f-stable multicurve Γ to which γ belongs.

Let $A = -\log(2\log(\sqrt{2} + 1))$, so that any two distinct closed geodesics γ_1, γ_2 with $\omega(\tau, \gamma_i) > A$ are simple and disjoint. Hence there are at most $p - 3$ closed geodesics γ with $\omega(\tau, \gamma) > A$. Note then, that if $\omega(\tau) > B \coloneqq A + (p-3)m(\log d + 2D)$, there will be, in the "length spectrum" $\omega(\tau, \cdot)$ of τ after A, a "gap" of length at least $J \coloneqq m(\log d + 2D)$, that is, an interval such that no γ has this length. Let $[a, b]$ be the first such interval, where then $a \geq A$ and $b < B < \omega(\tau)$.

Lemma 0.2. *If $\Gamma_{J,\tau} := \{\gamma \subset (S^2, P_f) \colon \omega(\tau,\gamma) > b\}$, then $\Gamma_{J,\tau}$ is an f-stable multicurve. Moreover, the closed geodesics in $\mathbb{P}^1 \setminus P''$ of length $< d^m e^{-b}$ are exactly the components of $(f^m)^{-1}(\Gamma_{J,\tau})$.*

The lemma above is true because, as we saw above, the difference between $\omega(\tau,\gamma)$ and a $\omega(\tau,\gamma')$ for γ' a component of $(f^m)^{-1}(\gamma)$ cannot cross the gap of size J. A similar reasoning applies to show the second part. This also shows that if $\tilde{\gamma}$ is the shortest closed geodesic of τ', then either $l_{\tau'}(\tilde{\gamma}) \geq d^m e^{-b}$, so that

$$\omega(\tau') \leq b - m\log d < B - m\log d < \omega(\tau) - m\log d,$$

or $\tilde{\gamma}$ is a component of $(f^m)^{-1}(\Gamma_{J,\tau})$, and as $\Gamma_{J,\tau}$ is f-stable under the hypotheses, we have that $\tilde{\gamma}$ is either non-essential or homotopic to a curve in $\Gamma_{J,\tau}$.

Now suppose that, for some $\tau = \tau_i$, we have $\omega(\tau) > B$. Then we consider the f-stable multicurve $\Gamma = \Gamma_{J,\tau}$, which naturally contains the closed curve of shortest length in τ. If

$$v = \begin{bmatrix} l_\tau(\gamma_1)^{-1} \\ \vdots \\ l_\tau(\gamma_n)^{-1} \end{bmatrix}, \quad v' = \begin{bmatrix} l_{\tau'}(\gamma_1)^{-1} \\ \vdots \\ l_{\tau'}(\gamma_n)^{-1} \end{bmatrix}$$

we have that

$$f_\Gamma^m v = \begin{bmatrix} \sum_{j,\alpha} l_{\mathbb{P}^1\setminus P''}(\gamma_{1,j,\alpha})^{-1} \\ \vdots \\ \sum_{i,\alpha} l_{\mathbb{P}^1\setminus P''}(\gamma_{n,j,\alpha})^{-1} \end{bmatrix}.$$

We also know how to compare the length vectors $f_\Gamma^m v$ and v' through the analytic inclusion $\mathbb{P}^1 \setminus P'' \hookrightarrow \mathbb{P}^1 \setminus P'$ (proposition 7.1), which gives us that, for $1 \leq i \leq n$,

$$[v' - f_\Gamma^m v]_i \leq \frac{2}{\pi} + \frac{pd^m}{L_0} \leq \frac{2}{\pi} + pe^B,$$

where $L_0 = d^m e^{-B}$. This is because the components $\gamma_{i,j,\alpha}$ of $(f^m)^{-1}(\gamma_j)$ which are homotopic to γ_i are exactly those curves in $\mathbb{P}^1 \setminus P''$ homotopic to γ_i in $\mathbb{P}^1 \setminus P'$ of length less than $L = d^m e^{-b} < L_0$.

If $|\cdot|$ denotes the supremum norm of a vector in $\mathbb{R}^\Gamma$, then

$$|v'| < \frac{1}{2}|v| + \frac{2}{\pi} + pe^B,$$

and therefore

$$\exp(\omega(\tau')) < \frac{1}{2}\exp(\omega(\tau)) + \frac{2}{\pi} + pe^B.$$

The linear iteration $x \mapsto \frac{1}{2}x + \frac{2}{\pi} + pe^B$ has a unique attracting fixed point at $x = \frac{4}{\pi} + 2pe^B$, so that an asymptotic upper bound for $\omega(\sigma^i_{fm}(\tau_0))$ is $\log(\frac{4}{\pi} + 2pe^B)$.

Note that our choice of τ_0 is arbitrary, hence $\omega(\tau_0)$ can be arbitrarily large. But the upper bound by the linear iteration gives us that

$$\sup_i \omega(\sigma^i_{fm}(\tau_0)) \leq \max\{B,\ \omega(\tau_0),\ \log(\frac{4}{\pi} + 2pe^B)\}$$

where

$$\begin{aligned} B &= A + (p-3)J \\ &= -\log(2\log(\sqrt{2}+1)) + m(p-3)(\log d + 2D). \end{aligned}$$

In conclusion,

$$\inf_i \min_\gamma l_{\sigma^i_{fm}(\tau_0)}(\gamma) \geq \min\{e^{-B},\ \min_\gamma l_{\tau_0}(\gamma),\ \left(\frac{4}{\pi} + 2pe^B\right)^{-1}\},$$

where the "asymptotic upper bound" for the smallest closed geodesic under iteration is

$$\left(\frac{4}{\pi} + 2pe^B\right)^{-1}.$$