Extremal Width

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1 Definitions

Let U be a Riemann surface (possibly with boundary), and Γ be a collection of curves on U. Without great loss, we will always assume that all curves are locally rectifiable, in the sense that, on each sufficiently small holomorphic chart, segments of the curve are mapped to rectifiable curves on $\mathbb{C}$. There are two relevant conformal invariants associated to Γ, the *extremal length* and *extremal width*, which are inverses of each other; the purpose of these notes is to define them and describe some relations they satisfy.

Definition 1.1. A *measurable conformal metric* μ on U corresponds to a collection of non-negative measurable functions $\mu(z)$ such that on each chart μ is expressed as $\mu = \mu(z)|dz|$, so the relevant transformation rules with respect to change of coordinates hold:

$$\mu = \mu(z)|dz| = \mu(z(w))|z'(w)||dw|.$$

More abstractly, a measurable conformal metric can be defined as a measurable function $\mu : T^1U \to \mathbb{C}$ on the tangent bundle of U such that, if v is a tangent vector at some point p and $\lambda \in \mathbb{C}$, then $\mu(\lambda v) = |\lambda|\mu(v)$.

We will mostly deal with the first notion. Even though we really only consider measurable conformal metrics up to sets of measure 0, we will commit the slight abuse of notation in ignoring this nuance. Hereafter, every metric we refer to will be measurable and conformal, unless explicitly said otherwise.

Such metrics can be added together and scaled by non-negative constants, and they form a partial order by defining $\mu_1 \leq \mu_2$ if for all $z \in U$, $\mu_1(z) \leq \mu_2(z)$. Note that this does not depend on the conformal coordinate chosen. This allows us, for instance, to consider the metric given by the supremum or infimum of a collection of metrics.

Each μ defines an area element $\mu(z)^2|dz|^2 = \mu(z)^2 dz\overline{dz}$, which allows us to define area integrals of real and complex valued measurable functions on U. In particular, the *μ-area* of U is

$$A_\mu(U) \coloneqq \iint_U \mu^2 = \iint_U \mu(z)^2|dz|^2.$$

Given a curve γ on U, the *μ-length* $l_\mu(\gamma)$ of γ is defined as the line integral

$$l_\mu(\gamma) \coloneqq \int_\gamma \mu(z)|dz|$$

whenever μ is measurable with respect to γ as a function of arc-length, which is the case if it locally rectifiable. If Γ is a path family, let

$$l_\mu(\Gamma) = \inf_{\gamma\in\Gamma} l_\mu(\gamma).$$

Definition 1.2. The *extremal length* $\mathcal{L}(\Gamma)$ of Γ is defined as the quantity

$$\mathcal{L}(\Gamma) \coloneqq \sup_\mu \frac{l_\mu(\Gamma)^2}{A_\mu(U)},$$

where the supremum ranges over all measurable conformal metrics on U whose area is not 0 or ∞. Note that metrics that are scalar multiples of each other give the same ratio on the right, so that we could take the supremum over all metrics of area 1 (or some other homogeneous normalization). We could therefore talk about *projective measurable conformal metrics*, but we will not delve deeper into these notions.

Definition 1.3. The *extremal width* $\mathcal{W}(\Gamma)$ is defined as

$$\mathcal{W}(\Gamma) \coloneqq \inf_\mu \{A_\mu(U) \mid \forall\gamma \in \Gamma,\ l_\mu(\gamma) \geq 1\}.$$

Similarly, the infimum above could be taken over all metrics μ such that $l_\mu(\Gamma) \geq 1$, but in practice the definition above will be the most useful. It is straightforward to check that $\mathcal{W}(\Gamma) = \mathcal{L}(\Gamma)^{-1}$, and that these are conformal invariants; that is, if $f : U \to V$ is a biholomorphism, then $\mathcal{W}(f(\Gamma)) = \mathcal{W}(\Gamma)$.

Example 1.4. Consider the rectangle $R = [0, u] \times [0, 1]$ as a subset of $\mathbb{C}$ with the induced conformal structure, and let Γ be family of (locally rectifiable) paths connecting the top and bottom sides $[0, w] \times \{1\}$ and $[0, w] \times \{0\}$. By taking the standard euclidean metric on (the interior of) R, we get from the infimum definition that $\mathcal{W}(\Gamma) \leq w$. Now let μ be an arbitrary measurable conformal metric on R such that $l_\mu(\gamma) \geq 1$ for $\gamma \in \Gamma$. In particular, for all vertical paths we have

$$\int_0^1 \mu(x_0, y)dy \geq 1,$$

and integrating over all vertical paths we get

$$\int_0^w \int_0^1 \mu(x,y)dydx = \iint_R \mu dxdy \geq w.$$

By Cauchy-Schwarz:

$$w \leq \iint_R (\mu \cdot 1)dxdy \leq \left(\iint \mu^2\right)^{1/2} \left(\iint 1\right)^{1/2} = A_\mu(R)^{1/2} w^{1/2},$$

which reduces to $w \leq A_\mu(R)$. As μ was arbitrary, $w \leq \mathcal{W}(\Gamma)$, and so $\mathcal{W}(\Gamma) = w$. Note that the extremal width would be the same if we restricted to the family of only vertical paths of R, and that in either case the measurable conformal metric that realizes the infimum is the standard euclidean metric.

Example 1.5. Let A be the standard round annulus of inner radius r and outer radius R centered at the origin, and Γ be the collection of paths connecting the inner boundary to the outer one. Consider the metric $|dz/z|$ on A; its area is

$$\iint_A \frac{|dz|^2}{|z|^2} = \int_0^{2\pi} \int_r^R \frac{1}{\rho^2} \rho d\rho d\theta = 2\pi \log(R/r),$$

and the length of a path connecting the boundaries is at least the length of a radial path, which is

$$\int_r^R \frac{1}{\rho} d\rho = \log(R/r).$$

Rescaling the metric by $\log(R/r)^{-1}$ so that $l_\mu(\Gamma) = 1$, we get

$$\mathcal{W}(\Gamma) \leq \frac{2\pi}{\log(R/r)}.$$

A very similar argument using Cauchy-Schwarz's inequality gives

$$\mathcal{W}(\Gamma) = \frac{2\pi}{\log(R/r)} = \frac{1}{\operatorname{mod}(A)},$$

where $\operatorname{mod}(A)$ is the usual modulus of an annulus. Another point of view for this example is that by slitting the annulus along some radial line, the exponential map takes the rectangle $[\log r, \log R] \times [0, 2\pi]$ conformally onto the slit annulus, but paths connecting the boundaries of A correspond to paths connecting the left and right sides of the rectangle. It is not hard to check that extremal width will be the inverse of that of the path family connecting the top and bottom sides.

2 Parallel and series laws

Given distinct path families satisfying certain properties, one may compare their extremal lengths and widths. If Γ and Γ' are such that every curve $\eta \in \Gamma'$ contains some $\gamma \in \Gamma$ as a segment, we say that Γ' *overflows* Γ, and denote this by $\Gamma \preceq \Gamma'$. A straightforward consequence is that $\mathcal{L}(\Gamma) \leq \mathcal{L}(\Gamma')$, or equivalently $\mathcal{W}(\Gamma) \geq \mathcal{W}(\Gamma')$. This is because every metric μ such that $l_\mu(\gamma) \geq 1$ for all $\gamma \in \Gamma$ also satisfies $l_\mu(\eta) \geq 1$ for all $\eta \in \Gamma'$, so that in the formula for $\mathcal{W}(\Gamma')$ we are taking an infimum over a larger set of conformal metrics, being therefore smaller. Intuitively, the Γ' are "longer and fewer" (and so more metrics satisfy the condition that their μ-lengths are large).

As a remark, if Γ and Γ' are two path families such that $\Gamma \subseteq \Gamma'$, then in fact $\Gamma \succeq \Gamma'$, and so $W(\Gamma) \leq W(\Gamma')$. This is evident in the example of a vertical foliation of a rectangle of width w and height 1 as a subset of the vertial foliation on the wider rectangle of width $w' \geq w$ and height 1.

Proposition 2.1 (Parallel law). *Suppose Γ_1 and Γ_2 are two path families. Then*

$$\mathcal{W}(\Gamma_1 \cup \Gamma_2) \leq \mathcal{W}(\Gamma_1) + \mathcal{W}(\Gamma_2).$$

If Γ_1 and Γ_2 have disjoint support, then

$$\mathcal{W}(\Gamma_1 \cup \Gamma_2) = \mathcal{W}(\Gamma_1) + \mathcal{W}(\Gamma_2).$$

Proof. Let μ_i be a conformal metric for which $l_{\mu_i}(\gamma) \geq 1$ for all $\gamma \in \Gamma_i$, for $i = 1, 2$. Letting $\mu = \max\{\mu_1, \mu_2\}$, we see that $l_\mu(\gamma) \geq 1$ for all $\gamma \in \Gamma_1 \cup \Gamma_2$, and so

$$\mathcal{W}(\Gamma_1 \cup \Gamma_2) \leq A_\mu(U) \leq A_{\mu_1}(U) + A_{\mu_2}(U).$$

as this holds for all μ_1 and μ_2 over which the infimum in the definition of extremal length is defined, we get the first inequality.

If the families are disjoint, then every metric μ such that $l_\mu(\gamma) \geq 1$ for all $\gamma \in \Gamma_1 \cup \Gamma_2$ can be written as a sum $\mu = \mu_1 + \mu_2$ with disjoint supports on Γ_1 amd Γ_2 respectively, so that

$$\mathcal{W}(\Gamma_1) + \mathcal{W}(\Gamma_2) \leq A_\mu(U),$$

and since this holds for all μ, we get the opposite inequality. □

For x, y positive real numbers, we denote their harmonic sum by

$$x \oplus y \coloneqq \frac{1}{\frac{1}{x} + \frac{1}{y}}$$

Proposition 2.2 (Series law)**.** *Suppose that* Γ_1 *and* Γ_2 *are disjoint path families and* Γ *overflows both* Γ_1 *and* Γ_2*. Then*

$$\mathcal{W}(\Gamma) \leq \mathcal{W}(\Gamma_1) \oplus \mathcal{W}(\Gamma_2).$$

Proof. We may equivalently show that $\mathcal{L}(\Gamma) \geq \mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2)$. Choosing a metric μ_i for Γ_i of finite non-zero area, we may scale it so that $l_\mu(\Gamma_i) = A_{\mu_i}(U)$. Hence, by considering $\mu = \mu_1 + \mu_2$, $l_\mu(\Gamma) \geq l_{\mu_1}(\Gamma_1) + l_{\mu_2}(\Gamma_2)$, and

$$\mathcal{L}(\Gamma) \geq \frac{l_\mu(\Gamma)^2}{A_\mu(U)} \geq \frac{(l_{\mu_1}(\Gamma_1) + l_{\mu_2}(\Gamma_2))^2}{A_{\mu_1}(U) + A_{\mu_2}(U)} = l_{\mu_1}(\Gamma_1) + l_{\mu_2}(\Gamma_2) = \frac{l_{\mu_1}(\Gamma_1)^2}{A_{\mu_1}(U)} + \frac{l_{\mu_2}(\Gamma_2)^2}{A_{\mu_2}(U)}.$$

As this holds independently of the choice of μ_1 and μ_2, we get the desired inequality. □

Remark. Both laws obviously genrealize to finite collections of path families. More precisely, if $\Gamma_1, \ldots, \Gamma_n$ are path families and $\Gamma \subseteq \bigcup_{i=1}^n \Gamma_i$ (or, more weakly, every Γ_i overflows Γ), then

$$\mathcal{W}(\Gamma) \leq \sum_{i=1} \mathcal{W}(\Gamma_i),$$

and if Γ' simultaneously overflows all Γ_i, then

$$\mathcal{W}(\Gamma') \leq \bigoplus_{i=1}^{n} \mathcal{W}(\Gamma_i).$$

3 Changes under mappings

3.1 Pullbacks and pushforwards of metrics

In what follows, all holomorphic maps are assumed to be non-constant, unless stated otherwise.

Definition 3.1. Let $f : U \to V$ be a holomorphic map so that $f(w) = z$, and $\nu = \nu(z)|dz|$ a (measurable, conformal) metric on V. The *pullback* $f^*\nu$ of ν is the metric on U given by

$$f^*\nu = \nu(f(w))|f'(w)||dw|.$$

In the tangent bundle definition, $(f^*\nu)(v) = \nu(df(v))$, where v is a tangent vector and df is the differential.

By construction, the map f is a local isometry between $(U, f^*\nu)$ and (V, ν), which implies that f preserves the lengths of curves. If η is a curve in U, its image/pushforward $f_*\eta = f \circ \eta$ satisfies

$$\int_{f_*\eta} \mu ds = \int_\eta (f^*\mu) ds,$$

or equivalently, $l_{f^*\nu}(\eta) = l_\eta(f_*\eta)$.

Remark. In many of the definitions and results that will follow, we will want $f : U \to V$ to be also proper, so that it is a branched covering map of some finite degree d. In this case, if $C \subset U$ is the set of critical points and $f(C) \subset V$ the set of critical values, C and $f(C)$ will be discrete, and $f : U \setminus f^{-1}(f(C)) \to V \setminus C$ is an unbranched covering map of degree d. Elements of the deck group can be extended to biholomorphisms of U by filling in the punctures [For81], and in particular if ν is a metric on V, the deck group will consist of isometries of $(U, f^*\nu)$. Moreover, as when dealing with metrics and path families we often do not care about measure zero sets, it is harmless to assume that a proper map is unbranched in most scenarios.

Proposition 3.2. *If $f : U \to V$ is a proper holomorphic map of degree d and ν is a metric on V, then*

$$A_{f^*\nu}(U) = d \cdot A_\nu(V).$$

Proof. Without loss, assume that f is unbranched. Let $(W_i)_{i\in I}$ be a open covering of V by evenly covered neighborhoods, and $(\rho_i)_{i\in I}$ a partition of unity subordinate to the W_i. The functions $\rho_i \circ f$ form a partition of unity on U subordinate to the open covering $(f^{-1}(W_i))_{i\in I}$, and so

$$\iint_U f^*\nu^2 = \sum_{i\in I} \iint_{f^{-1}(W_i)} (\rho_i \circ f) f^*\nu^2 = \sum_{i\in I} d \iint_{W_i} \rho_i \nu^2 = d \iint_V \nu^2,$$

since $f^{-1}(W_i)$ consists of exactly d components, each isometric to W_i by f with respect to the metrics $f^*\nu$ and ν, thereby preserving the area. $\square$

Definition 3.3. Let $f : U \to V$ be a proper holomorphic map of degree d. If $\mu = \mu(w)|dw|$ is a metric on U, the *pushforward* $f_*\mu$ is the metric on V locally given by

$$f_*\mu = (f_*\mu)(z)|dz| = \sum_{f(w)=z} \frac{\mu(w)}{|f'(w)|} dz,$$

where the sum is over all preimages w of z, when z is not a critical value of f.

If $z \in V$ is not a critical value and N is a neighborhood on which the d inverse branches $g_i : N \to N_i$ of f are well defined, we (locally) obtain

$$f_*\mu = \sum_{i=1}^{d} g_i^*\mu = \left(\sum_{i=1}^{d} \mu(g_i(z))|g_i'(z)| \right) |dz|.$$

This is a sufficient description, as the local forms patch together and we may ignore the set of critical values to define a measurable conformal metric. With respect to the tensorial definition, we have

$$(f_*\mu)(v) = \sum_{i=1}^{d} (g_i^*\mu)(v) = \sum_{i=1}^{d} \mu(dg_i(v)),$$

where v is a tangent vector on V not at a critical value.

Proposition 3.4. *Let $f : U \to V$ be a proper holomorphic map of degree d, and μ a metric on U.*

a) If γ is a curve on $V \setminus f(C)$, then

$$l_{f_*\mu}(\gamma) = \sum_{i=1}^{d} l_\mu(\widetilde{\gamma}_i),$$

where the $\widetilde{\gamma}_i$ are the d lifts of γ.

b) $A_{f_*\mu}(V) = A_\mu(U)$.

Proof. Again we may assume f is unbranched. The first statement is true if γ is contained in a evenly covered open set of the unbranched covering, since adding up the lengths of the lifts corresponds to taking the pushforward metric on V. Covering the image γ by evenly covered open sets, dividing γ into segments such that each is contained in an element of the cover, and adding up the lengths, we get the desired formula.

For the second statement, we proceed as in Proposition 3.2 by taking a partition of unity on V subordinate to an open cover of evenly covered neighborhoods. The sum of the areas of each lifted neighborhood adds up to the area with respect to the pushforward metric. □

We also have statements describing how pushforwards and pullbacks interact with each other:

Proposition 3.5. *Let $f : U \to V$ be a proper holomorphic map of degree d. If ν is a metric on V, then*

$$f_*f^*\nu = d \cdot \nu.$$

Proof. This is immediate from the local form

$$f_* f^* \nu = \sum_{i=1}^{d} g_i^* f^* \nu = \sum_{i=1}^{d} (f \circ g_i)^* \nu = \sum_{i=1}^{d} \nu = d \cdot \nu.$$

$\square$

Proposition 3.6. *Let $f : U \to V$ be a proper holomorphic map of degree d and μ a metric on U. Then*

$$f^* f_* \mu \geq \mu.$$

Moreover, if the branched covering f is normal, that is, the deck group acts transitively on the fibers, then

$$f^* f_* \mu = \sum_{g \in \mathrm{Deck}(U/V)} g^* \mu.$$

Proof. From the local definitions,

$$f^* f_* \mu = f^* \left(\sum_{i=1}^{d} (g_i)^* \mu \right) = \sum_{i=1}^{d} (g_i \circ f)^* \mu.$$

On a neighborhood of $z \in U$, one of the compositions $g_i \circ f$ will be the identity, so that $f^* f_* \mu \geq \mu$. Assuming that f is a normal branched covering and knowing that U is path connected, the action of $\mathrm{Deck}(U/V)$ on fibers (and therefore on small neighborhoods of the points in a fiber) is free and transitive. Each group element will correspond to projecting by f and choosing some local lift g_i, giving the desired result. $\square$

3.2 Pullbacks and pushforwards of path families

If Γ is a path family on V, we may define its *pullback* $f^*\Gamma$ as the collection of lifts of curves in Γ, that is, the curves $\widetilde{\gamma}$ in U such that $f \circ \widetilde{\gamma} = \gamma \in \Gamma$. This definition is most useful when f is a proper, so that all curves avoiding the critical values of f are liftable.

Proposition 3.7. *if $f : U \to V$ is a proper map of degree d and Γ is a path family on V, then*

$$\mathcal{W}(f^*\Gamma) = d\,\mathcal{W}(\Gamma).$$

Proof. Let ν be a metric in V such that $l_\nu(\gamma) \geq 1$ for all $\gamma \in \Gamma$ and $f^*\nu$ its pullback. As $(U, f^*\nu), \to (V, \nu)$ is a local isometry, preserving lengths of curves, and the pullback multiplies areas by d, we readily conclude that

$\mathcal{W}(f^*\Gamma) \leq d\,\mathcal{W}(\Gamma)$. Now consider μ a metric on U such that, for every lift $\widetilde{\gamma}$ of a curve γ in Γ, $l_\mu(\widetilde{\gamma})$. By 3.4, $l_{f^*\mu}(\gamma) \geq d$ for all $\gamma \in \Gamma$ and $A_{f_*\mu}(V) = A_\mu(U)$. Rescaling the metric to $\frac{1}{d} f_*\mu$, we get the other inequality $\mathcal{W}(\Gamma) \leq d\,\mathcal{W}(f^*\Gamma)$. □

If Γ is a path family on U and $f : U \to V$ is holomorphic, then the *pushforward* $f_*\Gamma \coloneqq f(\Gamma) = \{f \circ \gamma \mid \gamma \in \Gamma\}$ is naturally a path family in V.

Proposition 3.8. *If $f : U \to V$ is a proper holomorphic map of degree d and Γ is a path family on U, then*

$$\frac{1}{d}\mathcal{W}(\Gamma) \leq \mathcal{W}(f(\Gamma)) \leq \mathcal{W}(\Gamma).$$

Proof. Let μ be a metric on U such that $l_\mu(\gamma) \geq 1$ for all $\gamma \in \Gamma$ and $f_*\mu$ its pushforward. Then $l_{f_*\mu}(f \circ \gamma) \geq l_\mu(\gamma) \geq 1$ for all $f \circ \gamma \in f(\Gamma)$, and so

$$\mathcal{W}(f(\Gamma)) \leq A_{f_*\mu}(V) = A_\mu(U).$$

By taking infima, we have $\mathcal{W}(f(\Gamma)) \leq \mathcal{W}(\Gamma)$. The other inequality follows from the fact that $\Gamma \subseteq f^* f(\Gamma)$ and Proposition 3.7. □

Remark. In the above statement, if we instead assume that the degree of f is *at most* d, we still get the same inequalities.

4 A special family

This brief section is a retelling of some known facts contained in [KL09]. Let U be a Riemann surface of finite type with boundary ∂U, and $K \subseteq \operatorname{int} U$ a compact set. We consider the path family $\Gamma(U, K)$ of all paths connecting K to ∂U, and $\operatorname{mod}(U, K) \coloneqq \mathcal{L}(\Gamma(U, K))$, $\mathcal{W}(U, K) \coloneqq \mathcal{W}(\Gamma(U, K))$. We first note that the family of paths Γ' *contained in* $U \setminus K$ and connecting K to ∂U has the same width, because $\Gamma' \subseteq \Gamma$ and Γ overflows Γ'.

Proposition 4.1. *Let $f : U \to V$ be a proper holomorphic map of degree d between Riemann surfaces with boundary, $A \subseteq \operatorname{int} U$ compact and $B = f(A) \subseteq \operatorname{int} V$. Then*

$$\operatorname{mod}(U, A) \leq \operatorname{mod}(V, B) \leq d \cdot \operatorname{mod}(U, A).$$

Proof. If $\Gamma = \Gamma(U, A)$ and $\Gamma' = (V, B)$, then $f(\Gamma) = \Gamma'$ as f is proper. The inequality is an immediate consequence of 3.8. □

References

[Ahl06] L. Ahlfors. *Lectures on Quasiconformal Mappings*. 2nd ed. Vol. 38. University Lecture Series. American Mathematical Society, 2006.

[For81] O. Forster. *Lectures on Riemann Surfaces*. Graduate Texts in Mathematics 81. Springer-Verlag New York, 1981.

[KL09] J. Kahn and M. Lyubich. “The Quasi-Additivity Law in conformal geometry”. In: *Annals of Mathematics* 169 (2009), pp. 561–593.